Section 13.3, Problem 58

The Triangle Inequality for vectors is

$$|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}| \tag{1}$$

Part (a): give a geometric interpretation of the Triangle Inequality.

By the defintion of vector addition, if $|\vec{a}|$ and $|\vec{b}|$ are two sides of a triangle, where the tip of $\vec{a}$ is at the tail of $\vec{b}$, then $|\vec{a}+\vec{b}|$ is the length of the third side of this triangle. So the triangle inequality states that the length of the third side is less than the sum of the lengths of the other two sides. This is a classical theorem of Euclidean Geometry, written in terms of vectors.

Part (b): Use the Cauchy–Schwarz inequality $|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$ to prove the Triangle inequality.

Following the hint, we consider

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} \tag{2}$$

Since $\vec{a}\cdot\vec{b} \leq |\vec{a}\cdot\vec{b}|$ (any number is less than or equal to its absolute value)

$$|\vec{a}+\vec{b}|^2 \leq |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}\cdot\vec{b}| \tag{3}$$

Using Cauchy–Schwarz

$$|\vec{a}+\vec{b}|^2 \leq |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}| \tag{4}$$

But now

$$|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}| = (|\vec{a}|+|\vec{b}|)^2 \tag{5}$$

Hence

$$|\vec{a}+\vec{b}|^2 \leq (|\vec{a}|+|\vec{b}|)^2 \tag{6}$$

On both sides of this inequality, the quantity under the square is known to be positive, so we can remove the squares.

$$|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}| \tag{7}$$