Section 11.4, Problem 38

Find all points of intersection of the two curves

$$r = 1 - \cos\theta, \quad r = 1 + \sin\theta \tag{1}$$

The first thing to do is graph the curves: we find that $r = 1 - \cos\theta$ is a cardioid with the "dimple" on the right side, while $r = 1 + \sin\theta$ is a cardioid with the dimple at the bottom. If the graph is precise enough one should be able to see 3 intersection points.

There are three ways that two pairs (r_1, θ_1) and (r_2, θ_2) can represent the same point in the plane:

1. The radii are equal: $r_1 = r_2$, and the angles differ by an even multiple of π, that is, $\theta_1 = \theta_2 + 2n\pi$ for some integer n.

2. The radii are opposite: $r_1 = -r_2$, and the angles differ by an odd multiple of π, that is, $\theta_1 = \theta_2 + (2n+1)\pi$.

3. The radii are both zero: $r_1 = 0 = r_2$. In this case both points are the origin.

To look for intersections of the first kind, we just set the two r-values equal (since sin and cos are 2π periodic, we don't have to worry about even multiples of π):

$$1 - \cos\theta = r = 1 + \sin\theta \tag{2}$$

$$-\cos\theta = \sin\theta \tag{3}$$

$$-1 = \tan\theta \tag{4}$$

This equation holds if $\theta = 3\pi/4$ or $7\pi/4$. The corresponding radii are $r(3\pi/4) = 1 + 1/\sqrt{2}$, and $r(7\pi/4) = 1 - 1/\sqrt{2}$. So we've found intersection points

$$(r, \theta) = (1 + 1/\sqrt{2}, 3\pi/4), \quad (1 - 1/\sqrt{2}, 7\pi/4) \tag{5}$$

There is also an intersection of the third kind, that is, at the origin. This happens when $\theta = 0$ in $r = 1 - \cos\theta$, and when $\theta = 3\pi/2$ in $r = 1 - \sin\theta$. So the origin is an intersection point.

These are all three intersection points, and we're done.

As an aside, we can ask why there are no intersection points of the second kind, where the θ values differ by π. Such would be found by solving

$$1 - \cos(\theta + \pi) = -(1 + \sin\theta) \tag{6}$$

Using $\cos(\theta + \pi) = -\cos\theta$

$$1 + \cos\theta = -1 - \sin\theta \tag{7}$$

$$\cos\theta + \sin\theta = -2 \tag{8}$$

But this never happens: $\cos\theta$ and $\sin\theta$ are always at least -1, and for this equation to hold, we would need both to be -1 for the same θ, which is impossible (by, e.g., the pythagorean identity).