Section 11.2, Problem 53

Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$ (we assume $a > b > 0$) is

$$L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\,d\theta \tag{1}$$

where $e = \frac{1}{a}\sqrt{a^2 - b^2}$ is the eccentricity of the ellipse.

The given parameterization of the ellipse makes one full rotation when θ goes from 0 to 2π. However, by dividing the ellipse into four congruent segments along the major and minor axes of the ellipse, we realize that the total length is equal to 4 times the length as θ goes from 0 to $\pi/2$. Thus

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta \tag{2}$$

It remains to get the integrand in the desired form. We have

$$\frac{dx}{d\theta} = a\cos\theta, \quad \frac{dy}{d\theta} = -b\sin\theta \tag{3}$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \tag{4}$$

Using $\cos^2\theta = 1 - \sin^2\theta$, we obtain

$$= \sqrt{a^2 - a^2\sin^2\theta + b^2\sin^2\theta} = \sqrt{a^2 - (a^2 - b^2)\sin^2\theta} \tag{5}$$

$$= \sqrt{a^2\left[1 - \left(\frac{a^2 - b^2}{a^2}\right)\sin^2\theta\right]} = a\sqrt{1 - e^2\sin^2\theta} \tag{6}$$

where we recognize $e^2 = \frac{a^2 - b^2}{a^2}$.

Thus

$$L = 4\int_0^{\pi/2} a\sqrt{1 - e^2\sin^2\theta}\,d\theta \tag{7}$$

and we can pull the constant a out of the integral.