Section 12.9, Problem 38

(a) Starting with the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

find the sum of the series

$$\sum_{n=1}^{\infty} nx^{n-1}$$

for $|x| < 1$.

We simply differentiate the geometric series:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

Because the geometric series converges for $|x| < 1$, this equation is valid for $|x| < 1$.

(b) Find the sum of the following series:

$$\sum_{n=1}^{\infty} nx^n, \ |x| < 1 \qquad \sum_{n=1}^{\infty} \frac{n}{2^n}$$

For the first one, we just multiply the result of part (a) by x:

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$$

The second sum is just the first one with $x = 1/2$ substituted for x. Note that since $|1/2| < 1$ the power series expansion is valid at this x-value. Thus

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{(1/2)}{(1-(1/2))^2} = 2$$

(c) Find the sum of the following series:

$$\sum_{n=2}^{\infty} n(n-1)x^n, \ |x| < 1, \qquad \sum_{n=2}^{\infty} \frac{n^2-n}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

For the first sum, differentiate the geometric series twice

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n = \frac{d^2}{dx^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}$$

And multiply by x^2:

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \cdot \sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2 \cdot \frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3}$$

This is valid in the same interval as the geometric series itself: $|x| < 1$.

For the second sum, we just substitute $x = 1/2$:

$$\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} = \sum_{n=2}^{\infty} n(n-1)(1/2)^n = \frac{2(1/2)^2}{(1-(1/2))^3} = 4$$

For the third sum, we need to use the previous line plus the fact from part (b) that

$$2 = \sum_{n=1}^{\infty} \frac{n}{2^n}$$

Now

$$4 = \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} = \sum_{n=1}^{\infty} \frac{n^2 - n}{2^n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n} - \sum_{n=1}^{\infty} \frac{n}{2^n}$$

In the first step, we changed the sum from starting at $n = 2$ to starting at $n = 1$: This is okay because the $n = 1$ term is actually zero.

Thus

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$$