Section 12.6, Problem 39

Given any series $\sum a_n$, we define a series $\sum a_n^+$ consisting of the positive terms of $\sum a_n$, and a series $\sum a_n^-$ consisting of the negative terms of $\sum a_n$. More specifically, we define

$$a_n^+ = \frac{a_n + |a_n|}{2} \quad a_n^- = \frac{a_n - |a_n|}{2} \tag{1}$$

Thus, if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, while if $a_n < 0$, then $a_n^+ = 0$ and $a_n^- = a_n$.

Part (a): If $\sum a_n$ is absolutely convergent, show that both of the series $\sum a_n^+$ and $\sum a_n^-$ are convergent.

Because $\sum a_n$ is absolutely convergent, we know that both $\sum a_n$ and $\sum |a_n|$ are convergent. Therefore

$$\sum a_n^+ = \sum \frac{a_n + |a_n|}{2} = \frac{1}{2}(\sum a_n + \sum |a_n|) \tag{2}$$

expresses $\sum a_n^+$ as a sum of two convergent series (divided by 2), and so $\sum a_n^+$ is convergent.

Similarly, $\sum a_n^- = \frac{1}{2}(\sum a_n - \sum |a_n|)$ is the difference of two convergent series, so $\sum a_n^-$ is convergent.

Part (b): If $\sum a_n$ is conditionally convergent, show that both series $\sum a_n^+$ and $\sum a_n^-$ are divergent.

For $\sum a_n$ to be conditionally convergent means that $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

Suppose that $\sum a_n^+$ were convergent. We can write

$$|a_n| = 2a_n^+ - a_n \tag{3}$$

And so $\sum |a_n| = \sum (2a_n^+ - a_n) = 2\sum a_n^+ - \sum a_n$. This represents $\sum |a_n|$ as the sum of two convergent series, forcing $\sum |a_n|$ to be convergent, and contradicting our assumption.

The other case is similar. Suppose $\sum a_n^-$ were convergent. Then using

$$|a_n| = a_n - 2a_n^- \tag{4}$$

we obtain that $\sum |a_n| = \sum a_n - 2\sum a_n^-$ is the difference of two convergent series, hence convergent, contradicting our assumption.