Section 12.4, problem 44

Show that if $a_n > 0$, and $\sum a_n$ is convergent, then $\sum \ln(1 + a_n)$ is convergent.

We apply the limit comparison test. Consider the limit

$$\lim_{n\to\infty} \frac{\ln(1 + a_n)}{a_n} \tag{1}$$

Now, because $\sum a_n$ is convergent, we know that $\lim_{n\to\infty} a_n = 0$ (the contrapositive of this statement is the "test for divergence"). Thus the limit in question is equal to the limit of the function

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = \lim_{x\to 0} \frac{1}{1 + x} = 1 \tag{2}$$

where we have used L'Hospital rule.

Thus $\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = 1$, which is finite and positive, so $\sum \ln(1 + a_n)$ converges because $\sum a_n$ does.

Another approach is to use the (ordinary) comparison test. This involves proving $\ln(1 + a_n) \leq a_n$. In fact, for $x > -1$,

$$\ln(1 + x) \leq x \tag{3}$$

This can be proved by considering the difference of the two sides:

$$f(x) = \ln(1 + x) - x \tag{4}$$

Then

$$f'(x) = \frac{1}{1 + x} - 1 \tag{5}$$

and $f'(x) = 0$ at $x = 0$, so $x = 0$ is the only critical point. Taking the second derivative:

$$f''(x) = \frac{-1}{(1 + x)^2} \tag{6}$$

Since the second derivative is negative for all $x > -1$, the critical point at $x = 0$ is in fact a global maximum. Thus $f(x) \leq 0$ for all $x > -1$.

Since $a_n > 0$, we have $\ln(1 + a_n) \leq a_n$, and the comparison test applies. So $\sum \ln(1 + a_n)$ is convergent.