Section 16.4, Problem 36

Part (a): If D_a is the disk of radius a centered at the origin

$$\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta \tag{1}$$

where we have written the integral in polar coordinates using $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$. This integral is straightforward to evaluate. Making the substitution $u = -r^2$, $du = -2r\,dr$, we have obtain

$$\int_0^{2\pi}\int_0^{-a^2} e^u \frac{du}{-2}\,d\theta = \int_0^{2\pi} \frac{1}{-2}\left[e^u\right]_0^{-a^2}\,d\theta = \int_0^{2\pi} \frac{1}{-2}(e^{-a^2}-1)\,d\theta = 2\pi\frac{1}{-2}(e^{-a^2}-1) = \pi(1-e^{-a^2}) \tag{2}$$

Thus $\iint_{D_a} e^{-(x^2+y^2)}\,dA = \pi(1-e^{-a^2})$. To obtain the improper integral over the entire plane, we take the limit as $a \to \infty$.

$$I = \iint_{\mathbf{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\pi(1-e^{-a^2}) = \pi \tag{3}$$

Part (b): The equivalent definition of the improper integral is as the limit as $a \to \infty$ of integrals over squares $S_a = \{(x,y) \mid -a \le x \le a, -a \le y \le a\}$. Clearly,

$$\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dx\,dy = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy \tag{4}$$

Since e^{-y^2} is constant with respect to x, we can pull it out of the inner integral:

$$\int_{-a}^{a}\left[\int_{-a}^{a} e^{-x^2}\,dx\right] e^{-y^2}\,dy \tag{5}$$

Since the expression in square brackets is constant with respect to y, we can pull it out of the dy-integral:

$$\int_{-a}^{a} e^{-x^2}\,dx \int_{-a}^{a} e^{-y^2}\,dy \tag{6}$$

Thus

$$I = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx\int_{-a}^{a} e^{-y^2}\,dy = \int_{-\infty}^{\infty} e^{-x^2}\,dx\int_{-\infty}^{\infty} e^{-y^2}\,dy \tag{7}$$

Since in part (a) we found $I = \pi$, we get

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx\int_{-\infty}^{\infty} e^{-y^2}\,dy = \pi \tag{8}$$

Part (c): Now we observe that both integrals in the equation above are actually equal: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy$, since x and y are just dummy variables. Thus

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi \tag{9}$$

Hence the integral in question is $\pm\sqrt{\pi}$. It's clear that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ is positive, since it is the integral of a positive function. Thus it equals $\sqrt{\pi}$:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \tag{10}$$

Part (d): Let $t = \sqrt{2}\,x$. Then $x^2 = t^2/2$, and $dt = \sqrt{2}\,dx$, and when we substitute:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-t^2/2}\,\frac{dt}{\sqrt{2}} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt \tag{11}$$

Moving the factor of $\sqrt{2}$ over to the other side and changing the dummy variable from t back to x gives

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi} \tag{12}$$

Note: The function $f(x) = (e^{-x^2/2})/\sqrt{2\pi}$ *is called the Gaussian or normal distribution, which is important in probability theory. This function has* $\int_{-\infty}^{\infty} f(x)\,dx = 1$*, which is required for any probability distribution. This requirement explains the importance of the factor* $\sqrt{2\pi}$*.*