Section 16.1, Problems 17

If f is a constant function, $f(x, y) = k$, and $R = [a, b] \times [c, d]$, show that $\iint_R f(x, y)\, dA = k(b - a)(d - c)$.

One simple way to do this is to use the fact that the double integral $\iint_R f(x, y)\, dA$ is the volume under the graph of f. Since f is constant, the 3–dimensional shape under the graph of f is a rectangular prism, with height k, width, $(b - a)$, and length $(d - c)$. Hence its volume is $k(b - a)(d - c)$.

Another way to argue uses the Riemann sum definition of integral. We divide the rectangle R into smaller rectangles of width $\Delta x = (b - a)/m$ and length $\Delta y = (d - c)/n$. Thus there are mn total small rectangles. The Riemann sum is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta x\, \Delta y$$

Since f is constant, $f(x_i^*, y_j^*) = k$ no matter what the sample point (x_i^*, y_j^*) is, so this Riemann sum reduces to

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} k\, \Delta x\, \Delta y &= k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta x\, \Delta y = k(mn)\Delta x\, \Delta y \\ &= k(mn) \frac{b - a}{m} \frac{d - c}{n} = k(b - a)(d - c) \end{aligned}$$

Since this sum is independent of m and n, taking the limit as $m \to \infty$ and $n \to \infty$ does nothing, and we have that $\iint_R f(x, y)\, dA = k(b - a)(d - c)$.