## Section 15.4, Problem 42

Suppose you need to know an equation of the tangent plane to a surface $S$ at the point $P(2, 1, 3)$. You don't have an equation for $S$ but you know that the curves

$$\vec{r}_1(t) = \langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \rangle \tag{1}$$
$$\vec{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle \tag{2}$$

both lie on $S$. Find an equation for the tangent plane at $P$.

Because we know two curves contained in the surface, we know at least two vectors tangent to the surface $S$, namely the velocity vectors $\vec{v}_1(t) = \frac{d}{dt}\vec{r}_1(t)$ and $\vec{v}_2(u) = \frac{d}{dt}\vec{r}_2(u)$ of these two curves.

$$\vec{v}_1(t) = \langle 3, -2t, -4 + 2t \rangle \tag{3}$$
$$\vec{v}_2(u) = \langle 2u, 6u^2, 2 \rangle \tag{4}$$

Since we are interested the point $P(2, 1, 3)$, we need to find the value of the parameter $t$ when $\vec{r}(t)$ passes through $P$. Clearly $t = 0$ works. Similarly, the path $\vec{r}_2(u)$ passes through $P$ when $u = 1$. Thus,

$$\vec{v}_1(0) = \langle 3, 0, -4 \rangle \tag{5}$$
$$\vec{v}_2(1) = \langle 2, 6, 2 \rangle \tag{6}$$

are vectors tangent to $S$ at the point $P(2, 1, 3)$.

We are asked for the equation of the tangent plane, which now means a plane containing the two vectors $\langle 3, 0, -4 \rangle$ and $\langle 2, 6, 2 \rangle$. For the normal vector to this plane we may take the cross-product of these two vectors.

$$\langle 3, 0, -4 \rangle \times \langle 2, 6, 2 \rangle = \langle 24, -14, 18 \rangle \tag{7}$$

Now we have the normal vector $\langle 24, -14, 18 \rangle$ to the plane and the point $P(2, 1, 3)$ on the plane, so an equation for the plane is

$$24(x - 2) - 14(y - 1) + 18(z - 3) = 0 \tag{8}$$

A simpler form of the same equation is

$$12x - 7y + 9z = 44 \tag{9}$$