Section 12.1, problem 68

For clarity, this solution is somewhat wordier than we expect the students to turn in.

The sequence $\{a_n\}$ is defined by $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$.

1. Prove $a_n < 2$. We proceed by induction. The base case is

$$a_1 = \sqrt{2} = 1.414\ldots < 2$$

which is true. Suppose that $a_n < 2$. Then we have

$$2 + a_n < 4$$
$$\sqrt{2 + a_n} < \sqrt{4} = 2$$

Since the left-hand side of this inequality is the formula for a_{n+1}, we have $a_{n+1} < 2$, and the induction step is complete.

2. Prove $\{a_n\}$ is increasing, which is to say $a_{n+1} \geq a_n$. To see this, first observe that the inequality

$$\sqrt{2 + x} \geq x$$

is valid in for x in the interval $-2 \leq x \leq 2$. This can be seen, for example, by drawing the graphs of both sides of the inequality. Because a_n is always positive, and, by step 1, less than 2, we know a_n always lies in this interval of validity, and

$$a_{n+1} = \sqrt{2 + a_n} \geq a_n$$

Steps 1 and 2 show that $\{a_n\}$ is increasing and bounded above, so the monotonic sequence theorem applies, and

$$\lim_{n\to\infty} a_n = L$$

exists as a finite number.

To find the limit, we take the limit of both sides of the equation $a_{n+1} = \sqrt{2 + a_n}$:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n\to\infty} a_n}$$
$$L = \sqrt{2 + L}$$
$$L = 2$$