Directional Derivative & Gradient

Last time I flubbed the last example
please see the revised notes online for
two ways to do it.

Recall partial derivative $f(x,y)$

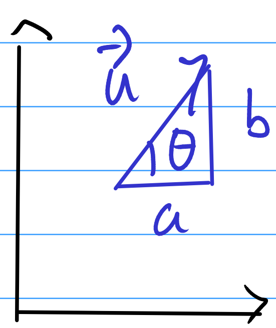
$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

"Derivative in the x-direction / $\vec{i}$-direction

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Derivative in the y-direction / $\vec{j}$-direction.

Can take derivative in __any__ direction



$\vec{u}$ is a unit vector
$\vec{u} = \langle a, b\rangle = \langle \cos\theta, \sin\theta\rangle$

$D_{\vec{u}} f(x_0, y_0)$ Directional derivative at (x_0,y_0) in direction $\vec{u}$

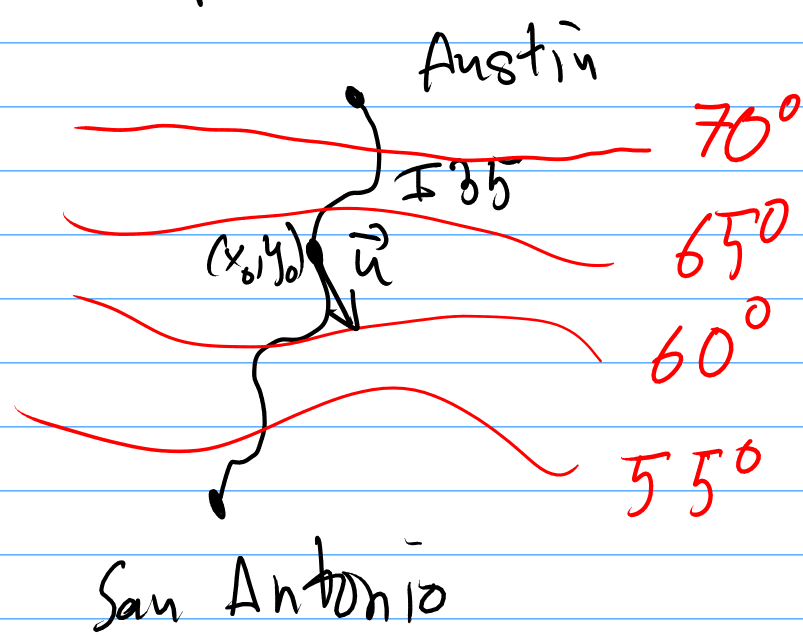
Start at (x_0, y_0), move in direction $\vec{u}$

$$\left.\begin{aligned} x &= x_0 + ha \\ y &= y_0 + hb \end{aligned}\right\} \text{ parametric equations of straight line}$$

Take derivative along this line

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Interpretation



$T(x,y)$ = temperature

$D_{\vec{u}} T(x_0, y_0)$

rate of change of T along my path.

As usual we don't work directly with the limit definition:

Can express direction derivative in terms of partial derivatives f_x, f_y

Given $f(x,y)$, (x_0, y_0), $\vec{u} = \langle a, b \rangle$

Trick: define a new function

$$g(h) = f(x_0 + ha,\ y_0 + hb)$$

$$g'(0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\vec{u}} f(x_0, y_0)$$

ON THE OTHER HAND

$g(h) = f(x,y)$ $\quad x = x_0 + ha,\ y = y_0 + hb$

$$\begin{tikzcd}[column sep=small] & g(h) \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & y \arrow[d, dash] \\ h & & h \end{tikzcd}$$

$$\frac{dg}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

$\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

Now recognize this as a dot product

$$= \underbrace{\langle f_x, f_y \rangle}_{} \cdot \underbrace{\langle a, b \rangle}_{\vec{u}}$$

This vector is called the <u>gradient</u> of f.

$$\text{grad } f = \nabla f = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$$
$$= \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

|| every directional derivative can be gotten from the gradient vector.

Ex $f(x,y) = \sin(2x+3y)$

$P = (-6,4) = (x_0, y_0)$

$\vec{u} = \frac{1}{2}(\sqrt{3}\,\vec{i} - \vec{j})$ (Note $|\vec{u}|=1$)

Compute gradient

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2\cos(2x+3y),$
$3\cos(2x+3y) \rangle$

Plug in point $(-6,4)$

$\nabla f(-6,4) = \langle 2\cos 0, 3\cos 0 \rangle = \langle 2,3 \rangle$

$D_{\vec{u}} f(-6,4) = \langle 2,3 \rangle \cdot \vec{u} = \langle 2,3 \rangle \cdot \frac{1}{2}\langle \sqrt{3}, -1 \rangle$
$= \sqrt{3} - \frac{3}{2}$

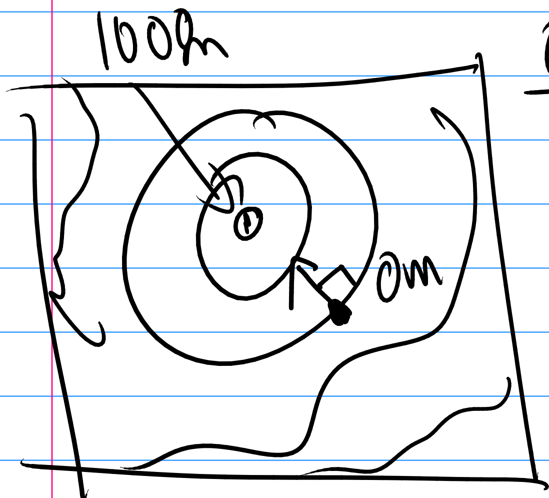
3 vars completely analogous

$f(x,y,z)$ $\vec{u} = \langle a,b,c \rangle$ $|\vec{u}| = 1$

$D_{\vec{u}} f(x_0,y_0,z_0) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle a,b,c \rangle$

$= \nabla f \cdot \vec{u}$

Suppose $f(x,y)$ represents height (as on a topographic map)

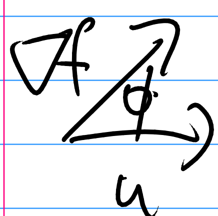


Q: what is the direction of steepest ascent?

A: the direction of ∇f.

$D_{\vec{u}} f$ = rate of ascent in the direction $\vec{u}$

$$\begin{aligned} D_{\vec{u}} f &= \nabla f \cdot \vec{u} = |\nabla f| \, |\vec{u}| \cos\phi \\ &= |\nabla f| \cos\phi \end{aligned}$$



greatest when $\phi = 0$

$\vec{u}$ and ∇f point in same direction

* If $\nabla f \perp \vec{u}$ then $D_{\vec{u}} f = \nabla f \cdot \vec{u} = 0$

This means that ∇f is perpendicular to the level sets of f

→ Tangent planes to level surfaces

$F(x,y,z)$ function of 3 vars

$F(x,y,z)=k$ level set

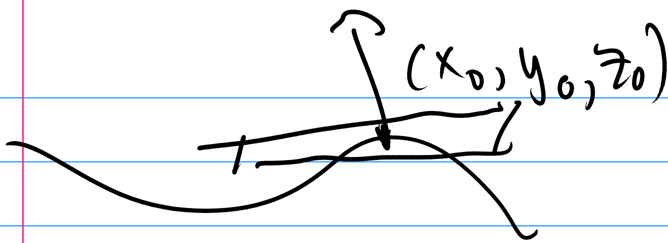
$\nabla F = \langle F_x, F_y, F_z \rangle$

suppose $\vec{u} = \langle a,b,c \rangle$ is a direction vector which is tangent to $\{F(x,y,z)=k\}$

$D_{\vec{u}} F = \nabla F \cdot \vec{u} = 0$

because moving along a level set does not change the value of F, by definition

If you want to know the normal vector to the tangent plane to a level set of F, you can use ∇F



equation of tangent plane

$$\nabla F \cdot \langle x - x_0,\ y - y_0,\ z - z_0 \rangle = 0$$

this is the
normal vector to the plane.