Functions as power series

Recall geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \boxed{\text{for } |x|<1}$$

we can regard this as a function of the variable x, and we've found a power series representation of $f(x) = \frac{1}{1-x}$.

<u>Importance</u>: Can work term by term. just like when you write numbers in decimal form, can work digit by digit.

$$\begin{array}{r} \overset{1}{1}05 \\ \times \quad 22 \\ \hline 210 \\ 2100 \\ \hline 2310 \end{array}$$

Using known power series to get others.

Ex
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Radius of convergence

$$R = 1$$

$$|-x^2| < 1$$
$$|x^2| < 1$$
$$|x| < 1$$

Ex
$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^2 \frac{1}{2\left(1+\frac{x}{2}\right)} = x^3 \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)}$$
$$= \frac{x^3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \frac{x^3}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{2^{n+1}}$$
$$= \sum_{n=3}^{\infty} (-1)^{n-3} \frac{x^n}{2^{n-2}} \quad \text{(reindexing)}$$

Valid if

$$\left|-\frac{x}{2}\right| < 1$$
$$|x| < 2$$
$$R = 2$$

Ex $\dfrac{1+x}{1-x} = (1+x)\dfrac{1}{1-x} = (1+x)\displaystyle\sum_{n=0}^{\infty} x^n$ $\boxed{\text{Need } |x|<1}$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} = \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} x^n$$

$$= (1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots)$$

$$= 1 + 2x + 2x^2 + 2x^3 + \cdots$$

$$= \boxed{1 + 2\sum_{n=1}^{\infty} x^n} = 1 + 2\frac{x}{1-x}$$

Great thing about power series

term-by-term differentiation & integration.

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Suppose the radius of convergence is $R > 0$.

So $f(x)$ is a function on the interval $(a-R, a+R)$

Then $f(x)$ is continuous & differentiable on $(a-R, a+R)$

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

And the radius of convergence is R (the same).

Ex $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$

$R = 1$

differentiate $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \qquad |x| < 1$$

$R = 1$

$$\int \frac{1}{1-x}\,dx = -\ln(1-x) + C$$

$$\int \sum_{n=0}^{\infty} x^n\,dx = \sum_{n=0}^{\infty} \int x^n\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C'$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} + C'$$

equate them,

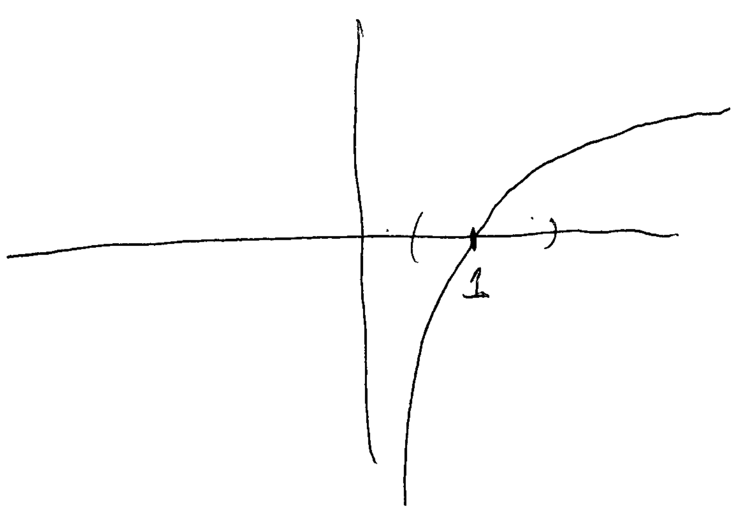
$$-\ln(1-x) + C = C' + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

plug in $x=0$: $0 + C = C' + 0$

take $C = C' = 0$

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x|<1$$

$$R = 1$$



Ex $\dfrac{1}{1+x^2}$ Find a powerseries for $\tan^{-1}(x)$.

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$ power series valid when $|x|<1$

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C'$$

$$\tan^{-1}(x) + C = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C'$$

plug $x=0$: $\tan^{-1}(0) = 0$ $\quad \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(0)^{2n+1}}{2n+1} = 0$

take $C = C' = 0$

$$\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} \quad \boxed{|x|<1}$$

$$\int \frac{\ln(1-t)}{t}\, dt$$

$$\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$$

$$\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$$

$$\int \frac{\ln(1-t)}{t}\, dt = -\sum_{n=1}^{\infty} \int \frac{t^{n-1}}{n}\, dt$$

$$= C + (-1)\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n} = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$$