NAME: SOLUTIONS

EID:

CIRCLE YOUR UNIQUE NUMBER:

54550 (8:30-9:30)   54555 (4:00-5:00)   54560 (5:00-6:00)

M408D Exam 3   Version B   November 18, 2011   James Pascaleff

Problem	Possible	Actual
1	15	
2	20	
3	20	
4	15	
5	15	
6	15	
7 (EC)	10 (EC)	
Total	100	

INSTRUCTIONS:

- Answer problems 1–6 for regular credit.
- Problem 7 is extra credit.
- Do all work on these sheets; use reverse side if necessary.
- Show all work.
- No books, notes, calculators, or other electronic devices.

1. (15 points) Consider the quadric surface defined by the equation

$$-\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$$

(a) (4 points) Describe the intersection of this quadric surface with a plane defined by an equation $x = a$, where a is a constant.

$$-\frac{a^2}{4} + \frac{y^2}{9} - z^2 = 1 \iff \frac{y^2}{9} - z^2 = 1 + \frac{a^2}{4}$$

hyperbola

(b) (4 points) Describe the intersection of this quadric surface with a plane defined by an equation $y = b$, where b is a constant. How does the shape depend on b?

$$-\frac{x^2}{4} + \frac{b^2}{9} - z^2 = 1 \iff \frac{x^2}{4} + z^2 = \frac{b^2}{9} - 1$$

if $b > 3$, ellipse ; if $b = 3$, point ;
if $b < 3$, empty

(c) (4 points) Describe the intersection of this quadric surface with a plane defined by an equation $z = c$, where c is a constant.

$$-\frac{x^2}{4} + \frac{y^2}{9} + c^2 = 1 \iff -\frac{x^2}{4} + \frac{y^2}{9} = 1 - c^2$$

hyperbola

(d) (3 points) Classify the quadric surface. Use a term like "ellipsoid" or "elliptic paraboloid," or similar. *Note:* the right answer is neither "ellipsoid" nor "elliptic paraboloid."

hyperboloid of two sheets

2. (20 points) Consider the curve given by

$$\vec{r}(t) = t^2\vec{i} + \cos(\pi t)\vec{j} + t^3 k$$

(a) (10 points) Find the tangent line to this curve at the point $(1, -1, 1)$. You can write the tangent line in any form you wish.

at $t=1$, $\vec{r}(1) = \langle 1, -1, 1 \rangle$

$\vec{r}\,'(t) = \langle 2t, -\pi\sin(\pi t), 3t^2 \rangle$, $\vec{r}\,'(1) = \langle 2, 0, 3 \rangle$

Tangent line: $\begin{cases} x(t) = 1 + 2t \\ y(t) = -1 \\ z(t) = 1 + 3t \end{cases}$

(b) (5 points) What is the speed of this curve as a function of t?

speed $= |\vec{r}\,'(t)| = \sqrt{(2t)^2 + (-\pi\sin(\pi t))^2 + (3t^2)^2}$

$= \sqrt{4t^2 + 9t^4 + \pi^2\sin^2(\pi t)}$

(c) (5 points) Set up, but *do not evaluate*, an integral that represents the arclength of this curve between $t = 0$ and $t = 1$.

arclength $= \int \text{speed}\, dt$

$= \int_0^1 \sqrt{4t^2 + 9t^4 + \pi^2\sin^2(\pi t)}\; dt$

3. (20 points) Consider the function

$$f(x, y) = \tan^{-1}(x + 8y)$$

(a) (5 points) Find the gradient of f at the point $(25, -3)$

$$f_x = \frac{1}{1+(x+8y)^2} = \frac{1}{1+(25-24)^2} = \frac{1}{2} \qquad \nabla f = \left\langle \frac{1}{2}, 4 \right\rangle$$

$$f_y = \frac{1}{1+(x+8y)^2} \cdot 8 = \frac{8}{2} = 4$$

(b) (5 points) Find an equation for the tangent plane to the surface $z = f(x, y)$ at the point $(25, -3, \pi/4)$.

$$z - \frac{\pi}{4} = f_x(x-25) + f_y(y+3)$$

$$z - \frac{\pi}{4} = \frac{1}{2}(x-25) + 4(y+3)$$

(c) (5 points) Using linear approximation near the point $(25, -3)$, find an approximation for $f(25, -3.1)$.

$$f(25,-3.1) \approx \frac{\pi}{4} + \frac{1}{2}(25-25) + 4(-3.1+3)$$

$$= \frac{\pi}{4} + 0 + 4(-0.1) = \frac{\pi}{4} - 0.4$$

(d) (5 points) In what direction at the point $(25, -3)$ does the function $f(x, y)$ *decrease* the fastest?

Opposite to the direction of the gradient

Direction of $-\nabla f = \left\langle -\frac{1}{2}, -4 \right\rangle$

4. (15 points) Find and classify the critical points of the function

$$f(x, y) = 4 + 2x + 8y - 3x^2 - 3y^2$$

$\frac{\partial f}{\partial x} = 2 - 6x = 0 \Rightarrow x = \frac{1}{3}$

$\frac{\partial f}{\partial y} = 8 - 6y = 0 \Rightarrow y = \frac{4}{3}$

so $\left(\frac{1}{3}, \frac{4}{3}\right)$ is the only critical point

$\frac{\partial^2 f}{\partial x^2} = -6$

$\frac{\partial^2 f}{\partial y^2} = -6$

$\frac{\partial^2 f}{\partial x \partial y} = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (-6)(-6) - 0$
$= 36 > 0$

$\Rightarrow$ max or min

$\frac{\partial^2 f}{\partial x^2} = -6 < 0 \Rightarrow$ max

$\left(\frac{1}{3}, \frac{4}{3}\right)$ is a local max.

5. (15 points) Use the method of Lagrange multipliers to find the point where the minimum value of the function

$$f(x, y, z) = 2x^2 + y^2 + 2z^2 + 4$$

subject to the constraint

$$g(x,y,z) = 2x + 2y + z = 4$$

occurs.

$$\nabla f = \langle 4x, 2y, 4z \rangle$$

$$\nabla g = \langle 2, 2, 1 \rangle$$

$$\nabla f = \lambda \nabla g \iff \begin{cases} 4x = 2\lambda \\ 2y = 2\lambda \\ 4z = \lambda \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}\lambda \\ y = \lambda \\ z = \frac{1}{4}\lambda \end{cases}$$

use $2x + 2y + z = 4$

$$2\left(\tfrac{1}{2}\lambda\right) + 2(\lambda) + \left(\tfrac{1}{4}\lambda\right) = 4$$

$$\frac{13}{4}\lambda = 4 \quad \Rightarrow \lambda = \frac{16}{13}$$

$$x = \frac{1}{2}\lambda = \frac{8}{13} \qquad y = \lambda = \frac{16}{13} \qquad z = \frac{1}{4}\lambda = \frac{4}{13}$$

min occurs at $\left(\frac{8}{13}, \frac{16}{13}, \frac{4}{13}\right)$

6. (15 points) Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a curve in 3–dimensional space. Let $\vec{v}(t) = \frac{d\vec{r}}{dt}$ and $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$ denote the velocity and acceleration. Let $V(x, y, z)$ be a function of 3 variables.

   Suppose that the equation

$$\vec{a}(t) = -\nabla V(\vec{r}(t))$$

   holds for all t. **Prove** that

$$\frac{d}{dt}\left[\frac{1}{2}|\vec{v}(t)|^2 + V(\vec{r}(t))\right] = 0$$

   Physics explanation (not needed to solve the problem): If $\vec{r}(t)$ represents the motion of a particle with mass $m = 1$, and $V(x, y, z)$ is the potential energy function, then the expression $\frac{1}{2}|\vec{v}(t)|^2 + V(\vec{r}(t))$ is the total energy of the particle, and $\vec{a}(t) = -\nabla V(\vec{r}(t))$ is the equation of motion (Newton's second law). So you are proving the law of conservation of energy.

SAME AS VERSION A

7. (10 points Extra Credit) Does this function have a limit at $(0, 0)$? Prove your answer.

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

SAME AS VERSION A.