NAME: SOLUTIONS

EID:

CIRCLE YOUR UNIQUE NUMBER:

54550 (8:30-9:30)   54555 (4:00-5:00)   54560 (5:00-6:00)

M408D Exam 3   Version A   November 18, 2011   James Pascaleff

INSTRUCTIONS:

- Answer problems 1–6 for regular credit.
- Problem 7 is extra credit.
- Do all work on these sheets; use reverse side if necessary.
- Show all work.
- No books, notes, calculators, or other electronic devices.

Problem	Possible	Actual
1	15	
2	20	
3	20	
4	15	
5	15	
6	15	
7 (EC)	10 (EC)	
Total	100	

1. (15 points) Consider the quadric surface defined by the equation

$$\frac{x^2}{4} - \frac{y^2}{9} - z^2 = 1$$

(a) (4 points) Describe the intersection of this quadric surface with a plane defined by an equation $x = a$, where a is a constant. How does the shape depend on a?

$$\frac{a^2}{4} - \frac{y^2}{9} - z^2 = 1 \iff \frac{y^2}{9} + z^2 = \frac{a^2}{4} - 1$$

if $|a| > 2$, ellipse; if $|a| = 2$, point;

if $|a| < 2$, empty.

(b) (4 points) Describe the intersection of this quadric surface with a plane defined by an equation $y = b$, where b is a constant.

$$\frac{x^2}{4} - \frac{b^2}{9} - z^2 = 1 \iff \frac{x^2}{4} - z^2 = 1 + \frac{b^2}{9}$$

hyperbola

(c) (4 points) Describe the intersection of this quadric surface with a plane defined by an equation $z = c$, where c is a constant.

$$\frac{x^2}{4} - \frac{y^2}{9} - c^2 = 1 \iff \frac{x^2}{4} - \frac{y^2}{9} = 1 + c^2$$

hyperbola

(d) (3 points) Classify the quadric surface. Use a term like "ellipsoid" or "elliptic paraboloid," or similar. *Note:* the right answer is neither "ellipsoid" nor "elliptic paraboloid."

hyperboloid of two sheets

2. (20 points) Consider the curve given by

$$\vec{r}(t) = t^2\vec{\imath} + t^4\vec{\jmath} + \cos(\pi t)\vec{k}$$

(a) (10 points) Find the tangent line to this curve at the point $(1, 1, -1)$. You can write the tangent line in any form you wish.

$t=1$, $\vec{r}(1) = \langle 1, 1, -1 \rangle$

$\vec{r}\,'(t) = \langle 2t, 4t^3, -\pi\sin(\pi t) \rangle$ at $t=1$, $\vec{r}\,'(1) = \langle 2, 4, 0 \rangle$

tangent line: $\begin{cases} x(t) = 1+2t \\ y(t) = 1+4t \\ z(t) = -1 \end{cases}$

(b) (5 points) What is the speed of this curve as a function of t?

speed $= |\vec{r}\,'(t)| = \sqrt{(2t)^2 + (4t^3)^2 + (-\pi\sin(\pi t))^2}$

$= \sqrt{4t^2 + 16t^6 + \pi^2\sin^2(\pi t)}$

(c) (5 points) Set up, but *do not evaluate*, an integral that represents the arclength of this curve between $t = 0$ and $t = 1$.

Arclength $= \int$ speed dt

$= \int_0^1 \sqrt{4t^2 + 16t^6 + \pi^2\sin^2(\pi t)}\, dt$

3. (20 points) Consider the function

$$f(x, y) = \tan^{-1}(4x + y)$$

(a) (5 points) Find the gradient of f at the point $(4, -15)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(4x+y)^2}(4) = \frac{1}{1+(4\cdot 4-15)^2}4 = \frac{1}{2}4 = 2$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(4x+y)^2}(1) = \frac{1}{2} \qquad \nabla f = \langle 2, \frac{1}{2} \rangle$$

(b) (5 points) Find an equation for the tangent plane to the surface $z = f(x, y)$ at the point $(4, -15, \pi/4)$.

$$z - \frac{\pi}{4} = \frac{\partial f}{\partial x}(x-4) + \frac{\partial f}{\partial y}(y+15)$$

$$z - \frac{\pi}{4} = 2(x-4) + \frac{1}{2}(y+15)$$

(c) (5 points) Using linear approximation near the point $(4, -15)$, find an approximation for $f(4.1, -15)$.

$$f(4.1, -15) \approx \frac{\pi}{4} + 2(4.1-4) + \frac{1}{2}(-15+15)$$

$$= \frac{\pi}{4} + 2(0.1) = \frac{\pi}{4} + 0.2$$

(d) (5 points) In what direction at the point $(4, -15)$ does the function $f(x, y)$ *decrease* the fastest?

the direction opposite to the gradient, that is, the direction of $-\nabla f$

$= \text{direction of } \langle -2, -\frac{1}{2} \rangle$

4. (15 points) Find and classify the critical points of the function

$$f(x, y) = 2 + 4x + 2y - 3x^2 - y^2$$

$$\frac{\partial f}{\partial x} = 4 - 6x = 0 \Rightarrow x = \frac{2}{3}$$

$$\frac{\partial f}{\partial y} = 2 - 2y = 0 \Rightarrow y = 1$$

so $\left(\frac{2}{3}, 1\right)$ is the only critical point.

$$\frac{\partial^2 f}{\partial x^2} = -6, \quad \frac{\partial^2 f}{\partial y^2} = -2 \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$D = (-6)(-2) - (0)^2 = 12 > 0$$

$\Rightarrow$ local max or min

$$\frac{\partial^2 f}{\partial x^2} = -6 < 0 \Rightarrow \text{local max}$$

$\left(\frac{2}{3}, 1\right)$ is a local max

5. (15 points) Use the method of Lagrange multipliers to find the point where the minimum value of the function

$$f(x, y, z) = 2x^2 + y^2 + 2z^2 + 3$$

subject to the constraint

$$g(x,y,z) = 2x + 3y + 2z = 5$$

occurs.

$$\nabla f = \langle 4x, 2y, 4z \rangle$$

$$\nabla g = \langle 2, 3, 2 \rangle$$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 4x = 2\lambda \\ 2y = 3\lambda \\ 4z = 2\lambda \end{cases} \Rightarrow \begin{array}{l} x = \frac{1}{2}\lambda \\ y = \frac{3}{2}\lambda \\ z = \frac{1}{2}\lambda \end{array}$$

using $2x + 3y + 2z = 5$

$$2\left(\tfrac{1}{2}\lambda\right) + 3\left(\tfrac{3}{2}\lambda\right) + 2\left(\tfrac{1}{2}\lambda\right) = 5$$

$$\Rightarrow \lambda + \frac{9}{2}\lambda + \lambda = 5$$

$$\Rightarrow \frac{13}{2}\lambda = 5 \Rightarrow \lambda = \frac{10}{13}$$

$$\Rightarrow x = \frac{5}{13}, \quad y = \frac{15}{13}, \quad z = \frac{5}{13}$$

so minimum occurs at

$$\left(\frac{5}{13}, \frac{15}{13}, \frac{5}{13}\right)$$

6. (15 points) Let $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a curve in 3–dimensional space. Let $\vec{v}(t) = \frac{d\vec{r}}{dt}$ and $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$ denote the velocity and acceleration. Let $V(x, y, z)$ be a function of 3 variables.

Suppose that the equation

$$\vec{a}(t) = -\nabla V(\vec{r}(t))$$

holds for all t. **Prove** that

$$\frac{d}{dt}\left[\frac{1}{2}|\vec{v}(t)|^2 + V(\vec{r}(t))\right] = 0$$

Physics explanation (not needed to solve the problem): If $\vec{r}(t)$ represents the motion of a particle with mass $m = 1$, and $V(x, y, z)$ is the potential energy function, then the expression $\frac{1}{2}|\vec{v}(t)|^2 + V(\vec{r}(t))$ is the total energy of the particle, and $\vec{a}(t) = -\nabla V(\vec{r}(t))$ is the equation of motion (Newton's second law). So you are proving the law of conservation of energy.

$$\frac{1}{2}|\vec{v}(t)|^2 = \frac{1}{2}\left(\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\right)^2$$

$$= \frac{1}{2}\left(x'(t)^2 + y'(t)^2 + z'(t)^2\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}|\vec{v}(t)|^2\right) = x'(t)\,x''(t) + y'(t)\,y''(t) + z'(t)\,z''(t)$$

$$= \vec{v}(t)\cdot\vec{a}(t)$$

Also,

$$\frac{d}{dt}\left(V(\vec{r}(t))\right) = \frac{d}{dt}\left[V(x(t), y(t), z(t))\right]$$

$$= \frac{\partial V}{\partial x}x'(t) + \frac{\partial V}{\partial y}y'(t) + \frac{\partial V}{\partial z}z'(t)$$

$$= \nabla V(\vec{r}(t))\cdot\vec{v}(t) = -\vec{a}(t)\cdot\vec{v}(t) \text{ by assumption}$$

Add together

$$\frac{d}{dt}\left[\frac{1}{2}|\vec{v}(t)|^2 + V(\vec{r}(t))\right] = \vec{v}(t)\cdot\vec{a}(t) - \vec{a}(t)\cdot\vec{v}(t) = 0$$

7. (10 points Extra Credit) Does this function have a limit at $(0, 0)$? Prove your answer.

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

Limit does not exist:

consider $x = t$, $y = 0$ $t \to 0$

$$\lim_{t \to 0} f(t, 0) = \frac{x0}{x^2 + 0^2} = 0$$

consider $x = t$, $y = t$ $t \to 0$

$$\lim_{t \to 0} f(t, t) = \frac{t^2}{t^2 + t^2} = \frac{1}{2}$$

since $\frac{1}{2} \neq 0$, the limits from two different directions do not agree, and

$\lim_{(x,y) \to (0,0)} f(x, y)$ Does not exist.