NAME: SOLUTIONS

EID:

CIRCLE YOUR UNIQUE NUMBER:

54550 (8:30-9:30)   54555 (4:00-5:00)   54560 (5:00-6:00)

M408D Exam 2   Version B   October 21, 2011   James Pascaleff

INSTRUCTIONS:

- Answer problems 1–6 for regular credit.
- Problem 7 is extra credit.
- Do all work on these sheets; use reverse side if necessary.
- Show all work.
- No books, notes, calculators, or other electronic devices.

Problem	Possible	Actual
1	20	
2	10	
3	20	
4	10	
5	20	
6	20	
7 (EC)	10 (EC)	
Total	100	

1. (a) (10 points) Find the degree 2 Taylor polynomial $T_2(x)$ for the function $f(x) = \sin(3x)$ centered at $x = 1$. [Note: an expression such as $\sin(3)$ cannot be simplified; feel free to leave it as it is.]

$f(x) = \sin(3x) \qquad f(1) = \sin(3)$

$f'(x) = 3\cos(3x) \qquad f'(1) = 3\cos(3)$

$f''(x) = -9\sin(3x) \qquad f''(1) = -9\sin(3)$

$$T_2(x) = \sin(3) + 3\cos(3)(x-1) - \frac{9\sin(3)}{2}(x-1)^2$$

(b) (10 points) Using Taylor's theorem and the fact that $|\cos(3x)| \leq 1$ for any x, find an upper bound for the remainder $|R_2(x)| = |f(x) - T_2(x)|$ of the degree 2 Taylor polynomial on the interval $|x-1| \leq \frac{1}{3}$. [Your answer should be an inequality of the form $|R_2(x)| \leq A$, where A is some definite number.]

$f'''(x) = -27\cos(3x)$

$|f'''(x)| = 27\,|\cos(3x)| \leq 27 \quad$ for $|x-1| \leq \frac{1}{3}$

So by Taylor's theorem

$$|R_2(x)| \leq \frac{27}{3!}|x-1|^3 \quad \text{for } |x-1| \leq \frac{1}{3}$$

$$\leq \frac{27}{3!}\left(\frac{1}{3}\right)^3 = \frac{27}{3!}\frac{1}{27} = \frac{1}{3!} = \frac{1}{6}$$

$$\underline{|R_2(x)| \leq \frac{1}{6}} \quad \text{for } |x-1| \leq \frac{1}{3}$$

2. (10 points) Using the first three terms of the power series for the logarithm

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

(that is, the $n = 1$, $n = 2$, and $n = 3$ terms), approximate the value of the integral

$$\int_0^1 \frac{\ln(1+x)}{x}\, dx$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\frac{\ln(1+x)}{x} \approx 1 - \frac{x}{2} + \frac{x^2}{3}$$

$$\int_0^1 \frac{\ln(1+x)}{x}\, dx \approx \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3}\right) dx$$

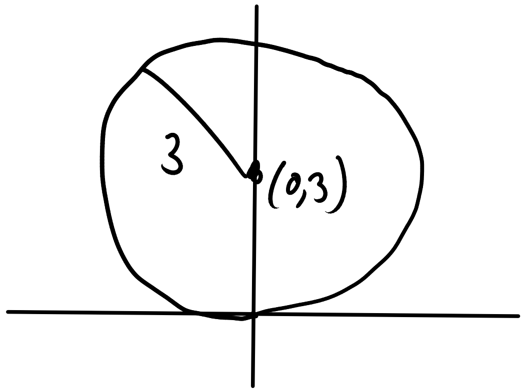
$$= \left[x - \frac{x^2}{2^2} + \frac{x^3}{3^2} \right]_0^1 = 1 - \frac{1}{2^2} + \frac{1}{3^2}$$

$$= 1 - \frac{1}{4} + \frac{1}{9}$$

3. Consider the circle given parametrically by

$$x(t) = 3\cos(t), \quad y(t) = 3\sin(t) + 3, \quad 0 \le t \le 2\pi$$

(a) (6 points) Find the center and radius of the circle. Draw a picture of it.



center = (0,3)

radius = 3

(b) (7 points) Using the derivatives $x'(t)$ and $y'(t)$, find the points where the curve has a vertical tangent line.

vertical tangent line $\iff x'(t) = 0$

$x'(t) = -3\sin t = 0 \Rightarrow t = 0, \pi$

at $t=0$, $(x,y) = (3,3)$; at $t=\pi$, $(x,y) = (-3,3)$

(c) (7 points) Using the integral formula for arclength, find the length of this curve.

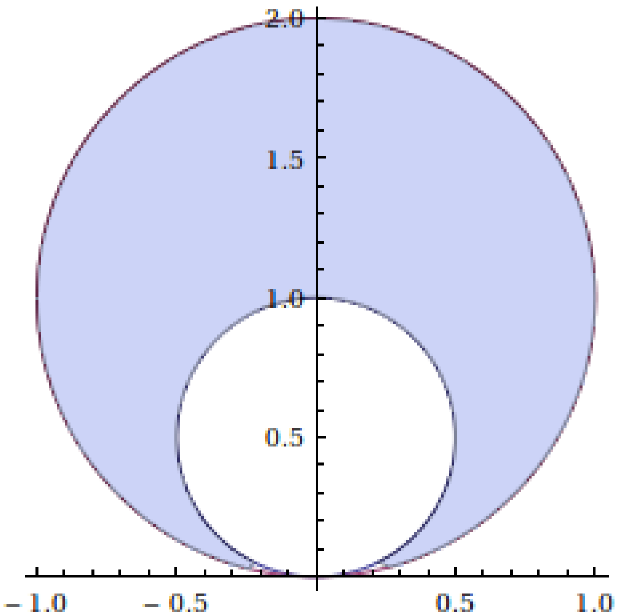
$$L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt = \int_0^{2\pi} \sqrt{9}\, dt$$

$$= 3\int_0^{2\pi} dt = 3 \cdot 2\pi = 6\pi$$

4. (10 points) Write down, but *do not evaluate* an integral that represents the area between the circles defined by the polar equations $r = 2\sin(\theta)$ and $r = \sin(\theta)$.



limits of integration

$\theta : 0 \to \pi$

$$\int_0^{\pi} \frac{1}{2}[2\sin\theta]^2\,d\theta - \int_0^{\pi} \frac{1}{2}[\sin\theta]^2\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\left([2\sin\theta]^2 - [\sin\theta]^2\right)d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(3\sin^2\theta)\,d\theta = \frac{3}{2}\int_0^{\pi} \sin^2\theta\,d\theta$$

Also correct $3\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$ (using symmetry)

5. Consider the vectors

$$\vec{a} = \vec{\imath} + 5\vec{\jmath} - 2\vec{k}$$
$$\vec{b} = 4\vec{\imath} - 2\vec{\jmath} - 3\vec{k}$$

(a) (10 points) Using the dot product, show that the angle between these vectors is $\pi/2$.

$$\vec{a} \cdot \vec{b} = 1 \cdot 4 + 5 \cdot (-2) + (-2)(-3)$$
$$= 4 - 10 + 6 = 0$$

so $|\vec{a}|\,|\vec{b}| \cos\theta = 0$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

(b) (10 points) Find the vector projection $\text{proj}_{\vec{a}}(\vec{a} + \vec{b})$

$$\text{proj}_{\vec{a}}(\vec{a}+\vec{b}) = \left[\frac{(\vec{a}+\vec{b}) \cdot \vec{a}}{|\vec{a}|^2}\right] \vec{a}$$

$$(\vec{a}+\vec{b}) \cdot \vec{a} = \underbrace{\vec{a} \cdot \vec{a}}_{\substack{\parallel \\ |\vec{a}|^2}} + \underbrace{\vec{b} \cdot \vec{a}}_{\substack{\parallel \\ 0 \text{ by part (a)}}} = |\vec{a}|^2$$

so $\text{proj}_{\vec{a}}(\vec{a}+\vec{b}) = \dfrac{|\vec{a}|^2}{|\vec{a}|^2}\vec{a}$

$$= \vec{a} = \vec{\imath} + 5\vec{\jmath} - 2\vec{k}$$

6. (20 points) Compute the volume of the parallelepiped with adjacent edges $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$, where

$$P = (0, 0, 3), \quad Q = (-1, 3, 0), \quad R = (2, 3, 0)$$

$\overrightarrow{OP} = \langle 0, 0, 3 \rangle$

$\overrightarrow{OQ} = \langle -1, 3, 0 \rangle$

$\overrightarrow{OR} = \langle 2, 3, 0 \rangle$

Volume = absolute value of $\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR})$

$$\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR}) = \begin{vmatrix} 0 & 0 & 3 \\ -1 & 3 & 0 \\ 2 & 3 & 0 \end{vmatrix} =$$

$$= 0 \begin{vmatrix} 3 & 0 \\ 3 & 0 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + 3 \begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= 3(-1 \cdot 3 - 2 \cdot 3) = 3(-9) = -27$$

Volume $= |-27| = 27$

7. (10 points extra credit) Prove the series expansion:

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots$$

SAME AS VERSION A