NAME: SOLUTIONS

EID:

CIRCLE YOUR UNIQUE NUMBER:

54550 (8:30-9:30) 54555 (4:00-5:00) 54560 (5:00-6:00)

M408D Exam 2 **Version A** **October 21, 2011** **James Pascaleff**

INSTRUCTIONS:

- Answer problems 1–6 for regular credit.
- Problem 7 is extra credit.
- Do all work on these sheets; use reverse side if necessary.
- Show all work.
- No books, notes, calculators, or other electronic devices.

Problem	Possible	Actual
1	20	
2	10	
3	20	
4	10	
5	20	
6	20	
7 (EC)	10 (EC)	
Total	100	

1. (a) (10 points) Find the degree 2 Taylor polynomial $T_2(x)$ for the function $f(x) = \sin(2x)$ centered at $x = 1$. [Note: an expression such as $\sin(2)$ cannot be simplified; feel free to leave it as it is.]

$f(x) = \sin(2x)$ $\quad$ $f(1) = \sin(2)$

$f'(x) = 2\cos(2x)$ $\quad$ $f'(1) = 2\cos(2)$

$f''(x) = -4\sin(2x)$ $\quad$ $f''(1) = -4\sin(2)$

$$T_2(x) = \sin(2) + 2\cos(2)(x-1) + \frac{(-4\sin(2))}{2}(x-1)^2$$

$$= \sin(2) + 2\cos(2)(x-1) - 2\sin(2)(x-1)^2$$

(b) (10 points) Using Taylor's theorem and the fact that $|\cos(2x)| \leq 1$ for any x, find an upper bound for the remainder $|R_2(x)| = |f(x) - T_2(x)|$ of the degree 2 Taylor polynomial on the interval $|x-1| \leq \frac{1}{2}$. [Your answer should be an inequality of the form $|R_2(x)| \leq A$, where A is some definite number.]

$f'''(x) = -8\cos(2x)$

$|f'''(x)| = 8|\cos(2x)| \leq 8$ for $|x-1| \leq \frac{1}{2}$

So by Taylor's theorem

$|R_2(x)| \leq \frac{8}{3!}|x-1|^3$ for $|x-1| \leq \frac{1}{2}$

$$\leq \frac{8}{3!}\left(\frac{1}{2}\right)^3 = \frac{8}{3!}\frac{1}{8} = \frac{1}{3!} = \frac{1}{6}$$

$\underline{|R_2(x)| \leq \frac{1}{6}}$ for $|x-1| \leq \frac{1}{2}$

2. (10 points) Using the first three terms of the power series for the inverse tangent

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

(that is, the $n = 0$, $n = 1$, and $n = 2$ terms), approximate the value of the integral

$$\int_0^1 \frac{\tan^{-1}(x)}{x}\, dx$$

$$\tan^{-1}(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5}$$

$$\frac{\tan^{-1}(x)}{x} \approx 1 - \frac{x^2}{3} + \frac{x^4}{5}$$

$$\int_0^1 \frac{\tan^{-1}(x)}{x}\, dx \approx \int_0^1 \left(1 - \frac{x^2}{3} + \frac{x^4}{5}\right) dx$$

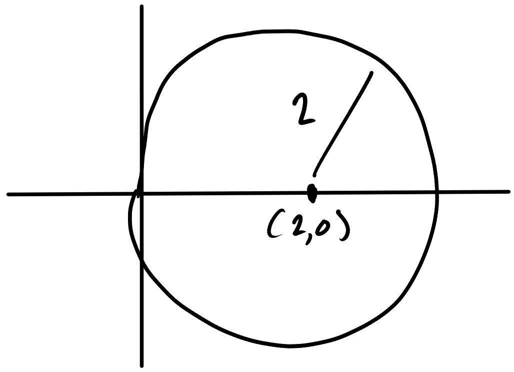
$$= \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} \right]_0^1 = 1 - \frac{1}{3^2} + \frac{1}{5^2}$$

$$= 1 - \frac{1}{9} + \frac{1}{25}.$$

3. Consider the circle given parametrically by

$$x(t) = 2\cos(t) + 2, \quad y(t) = 2\sin(t), \quad 0 \le t \le 2\pi$$

(a) (6 points) Find the center and radius of the circle. Draw a picture of it.



center = (2,0)

radius = 2

(b) (7 points) Using the derivatives $x'(t)$ and $y'(t)$, find the points where the curve has a horizontal tangent line.

horizontal tangent line $\Leftrightarrow$ $y'(t) = 0$

$y'(t) = 2\cos(t) = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

at $t = \frac{\pi}{2}$, $\underline{(x,y) = (2,2)}$; at $t = \frac{3\pi}{2}$, $\underline{(x,y) = (2,-2)}$

(c) (7 points) Using the integral formula for arclength, find the length of this curve.

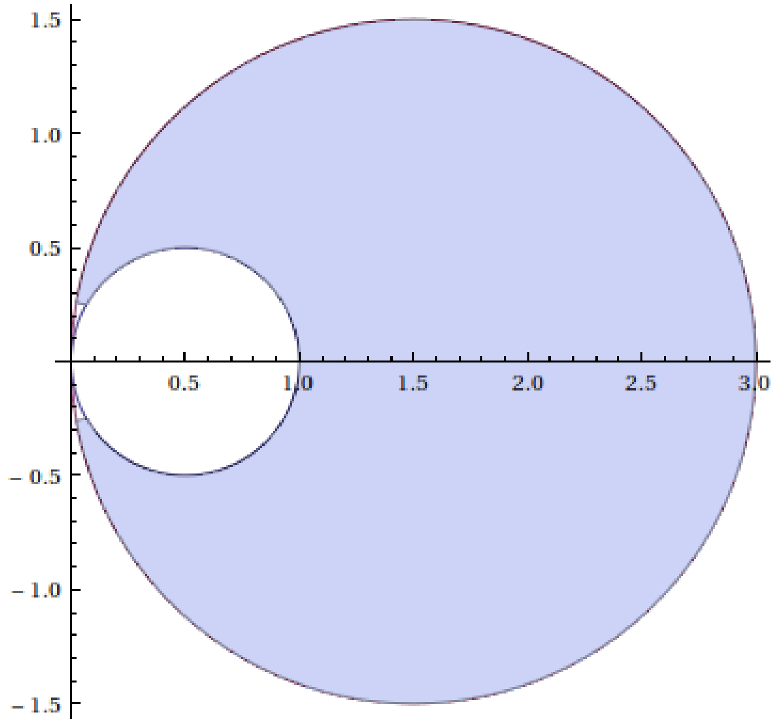
$$L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{4(\sin^2 t + \cos^2 t)}\, dt = \int_0^{2\pi} \sqrt{4}\, dt$$

$$= 2\int_0^{2\pi} dt = 2 \cdot 2\pi = 4\pi$$

4. (10 points) Write down, but *do not evaluate* an integral that represents the area between the circles defined by the polar equations $r = 3\cos(\theta)$ and $r = \cos(\theta)$.



limits of integration

$\theta: \frac{-\pi}{2} \to \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[3\cos\theta]^2\,d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}[\cos\theta]^2\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left([3\cos\theta]^2 - [\cos\theta]^2\right)d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(8\cos^2\theta\right)d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

ALSO correct: $8\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$ (using symmetry)

5. Consider the vectors

$$\vec{a} = 2\vec{i} + 5\vec{j} - 3\vec{k}$$

$$\vec{b} = 2\vec{i} - 2\vec{j} - 2\vec{k}$$

(a) (10 points) Using the dot product, show that the angle between these vectors is $\pi/2$.

$$\vec{a} \cdot \vec{b} = 2 \cdot 2 + 5 \cdot (-2) + (-3) \cdot (-2)$$

$$= 4 - 10 + 6 = 0$$

so $|\vec{a}||\vec{b}| \cos\theta = 0$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

(b) (10 points) Find the vector projection $\text{proj}_{\vec{a}}(\vec{a} + \vec{b})$

$$\text{proj}_{\vec{a}}(\vec{a} + \vec{b}) = \frac{[(\vec{a} + \vec{b}) \cdot \vec{a}]}{|\vec{a}|^2} \vec{a}$$

$$(\vec{a} + \vec{b}) \cdot \vec{a} = \underbrace{\vec{a} \cdot \vec{a}}_{|\vec{a}|^2} + \underbrace{\vec{b} \cdot \vec{a}}_{0 \text{ by part (a)}} = |\vec{a}|^2$$

so $\text{proj}_{\vec{a}}(\vec{a} + \vec{b}) = \frac{|\vec{a}|^2}{|\vec{a}|^2} \vec{a} =$

$$= \vec{a} = 2\vec{i} + 5\vec{j} - 3\vec{k}$$

6. (20 points) Compute the volume of the parallelepiped with adjacent edges $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$, where

$$P = (3, 0, 0), \quad Q = (0, -1, 3), \quad R = (0, 2, 3)$$

$\overrightarrow{OP} = \langle 3, 0, 0 \rangle$

$\overrightarrow{OQ} = \langle 0, -1, 3 \rangle$

$\overrightarrow{OR} = \langle 0, 2, 3 \rangle$

Volume = absolute value of $\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR})$

$$\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 2 & 3 \end{vmatrix} =$$

$$= 3 \begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= 3(-1 \cdot 3 - 2 \cdot 3) = 3 \cdot (-9) = -27$$

Volume $= |-27| = \underline{27}$

7. (10 points extra credit) Prove the series expansion:

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots$$

$$\sqrt{2} = (1+1)^{1/2}$$

Binomial series for $(1+x)^{1/2}$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \cdot 2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2 \cdot 3}x^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \cdots$$

plug in $x = 1$

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots$$