NAME: SOLUTIONS

EID:

CIRCLE YOUR UNIQUE NUMBER:

54550 (8:30-9:30)   54555 (4:00-5:00)   54560 (5:00-6:00)

M408D Exam 1 Version B **September 23, 2011** **James Pascaleff**

INSTRUCTIONS:

- Answer problems 1–6 for regular credit.
- Problem 7 is extra credit.
- Do all work on these sheets; use reverse side if necessary.
- Show all work.
- No books, notes, calculators, or other electronic devices.

Problem	Possible	Actual
1	10	
2	20	
3	10	
4	20	
5	20	
6	20	
7 (EC)	10 (EC)	
Total	100	

1. (10 points) Evaluate this limit by any method:

$$\lim_{x\to 1} \frac{1+x^5}{1+x^4}$$

You are not required to use L'Hospital's rule, but if you do, you must say whether this is an indeterminate form, and which indeterminate form it is.

$$\lim_{x\to 1} \frac{1+x^5}{1+x^4} = \frac{1+(1)^5}{1+(1)^4} = \frac{2}{2} = 1$$

NOT an indeterminate form

No L'Hospital's rule

2. For this problem, let

$$a_n = \frac{3n}{4n+7}$$

(a) (5 points) Consider

$$\lim_{n\to\infty} a_n$$

Does this limit exist? Evaluate it if it does exist.

$$\lim_{n\to\infty} \frac{3n}{4n+7} = \lim_{n\to\infty} \frac{3}{4+\frac{7}{n}} = \frac{3}{4}$$

Exists

(b) (5 points) Consider

$$\lim_{n\to\infty} (-1)^n a_n$$

Does this limit exist? Evaluate it if it does exist.

$\lim_{n\to\infty} (-1)^n \frac{3n}{4n+7}$ Does not exist

The sequence oscillates between $\frac{3}{4}$ and $-\frac{3}{4}$.

(c) (5 points) Is this series convergent?

$$\sum_{n=1}^{\infty} a_n$$

NO $\lim_{n\to\infty} a_n = \frac{3}{4} \neq 0$ Diverges by "test for divergence"

(d) (5 points) Is this series convergent?

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

No $\lim_{n\to\infty} (-1)^n a_n$ DNE diverges by "test for divergence".

3. (10 points) Is this series convergent? Evaluate the sum if it is convergent.

$$\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$$

Geometric series

initial term $a = 3\left(\frac{1}{4}\right)^0 = 3$

common ratio $r = \frac{1}{4}$

$|r| = \frac{1}{4} < 1$ so the series converges

$$\text{Sum} = \frac{a}{1-r} = \frac{3}{1-\frac{1}{4}} = \frac{3}{\left(\frac{3}{4}\right)} = 4$$

4. Consider the series

$$\sum_{n=3}^{\infty} \frac{4(\ln n)^3}{n}$$

Use the integral test to determine if it converges, in two steps:

(a) (8 points) Write down an integral you need to do in order to apply the integral test. That is, write down an improper integral whose convergence or divergence is equivalent to the convergence or divergence of the series.

$$\int_3^{\infty} \frac{4(\ln x)^3}{x}\, dx$$

(b) (7 points) List the properties of the function you are integrating that are needed in order for the integral test to apply. Don't check these properties, just list them. For example, we need the function to be continuous, and what else?

positive, continuous, decreasing

(c) (5 points) Determine whether the integral you wrote is convergent or divergent.

$$\int_3^{\infty} \frac{4(\ln x)^3}{x}\, dx = \lim_{b\to\infty} \int_3^{b} \frac{4(\ln x)^3}{x}\, dx$$

$$\left[u = \ln x \quad du = \frac{dx}{x}\right]$$

$$= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} 4u^3\, du = \lim_{b\to\infty} \left[u^4\right]_{\ln 3}^{\ln b}$$

$$= \lim_{b\to\infty} (\ln b)^4 - (\ln 3)^4 = (\infty)^4 - (\ln 3)^4 = \infty$$

DIVERGENT

5. Consider the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{4}{\sqrt{n}+5}$$

Determine whether the series is convergent (10 points), and whether it is absolutely convergent (10 points). In each case, state which tests you are using, and check that all conditions for those tests are satisfied. If you need to, you can take as given the fact that the sequence $\frac{4}{\sqrt{n}+5}$ is decreasing.

Alternating series test: $b_n = \dfrac{4}{\sqrt{n}+5}$

(1) b_n is decreasing ← Given

(2) $\lim_{n\to\infty} b_n = 0$

$$\lim_{n\to\infty} \frac{4}{\sqrt{n}+5} = \frac{4}{\infty+5} = \frac{4}{\infty} = 0 \quad \checkmark$$

so the series is convergent.

Absolute convergence: look at $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}+5}$

limit comparison w/ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (diverges by p-test)

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \Big/ \frac{4}{\sqrt{n}+5} = \lim_{n\to\infty} \frac{1}{4} \frac{\sqrt{n}+5}{\sqrt{n}} = \frac{1}{4} \lim_{n\to\infty} 1 + \frac{5}{\sqrt{n}} = \frac{1}{4}$$

Since $\frac{1}{4}$ is finite and not zero,

$\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}+5}$ diverges as well

NOT absolutely convergent.

6. Consider the power series centered at 1:

$$\sum_{n=1}^{\infty}(-1)^{n-1}n(x-1)^n$$

The interval of convergence of this power series is the set of all values of x for which the series is convergent. Determine the interval of convergence of the power series, in two steps:

(a) (10 points) Use the ratio test to determine the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-1|^{n+1}}{n\,\,|x-1|^n} = \lim_{n\to\infty}\frac{n+1}{n}|x-1|$$

$$= |x-1|\cdot\lim_{n\to\infty}\frac{n+1}{n} = |x-1|\cdot(1) = |x-1|$$

So $|x-1| < 1 \Rightarrow$ convergent

Radius of convergence $= 1$

(b) (10 points) Use other tests to check the endpoints of the interval of convergence.

Interval of convergence contains $(1-1, 1+1) = (0,2)$

At $x=0$, $\sum_{n=1}^{\infty}(-1)^{n-1}n(0-1)^n = \sum_{n=1}^{\infty}(-1)^{n-1}n(-1)^n$

$= \sum_{n=1}^{\infty}(-1)n \quad \lim_{n\to\infty}(-1)n = -\infty \neq 0$ <u>diverges</u> by test for divergence

At $x=2$ $\sum_{n=1}^{\infty}(-1)^{n-1}n(2-1)^n = \sum_{n=1}^{\infty}(-1)^{n-1}n$

$\lim_{n\to\infty}(-1)^{n-1}n$ DNE <u>diverges</u> by test for divergence

<u>Interval of convergence $= (0,2)$</u>

7. (EXTRA CREDIT 10 points) Determine the value of the constant C for which the following improper integral is convergent, and evaluate the integral for that value of C.

$$\int_0^1 \left(\frac{1}{x^3} - C\left(\frac{1}{x^3} - \frac{1}{\sqrt{x}}\right)\right) dx$$

This integral is improper at 0.

$$\lim_{t\to 0^+} \int_t^1 \left(\frac{1}{x^3} - \frac{C}{x^3} + \frac{C}{\sqrt{x}}\right) dx$$

$$= \lim_{t\to 0^+} \left[(1-C)\frac{-1}{2x^2} + 2C\sqrt{x} \right]_t^1$$

$$= \lim_{t\to 0^+} \left[(1-C)\frac{(-1)}{2} + 2C(1) - \underbrace{(1-C)\frac{(-1)}{2t^2}} - 2C\sqrt{t} \right]$$

In order for limit to exist, need to get rid of $\frac{1}{t^2}$ term

$\Rightarrow C = 1$

When $C=1$, it converges and equals

$$= \lim_{t\to 0^+} \left[0\,\frac{(-1)}{2} + 2\cdot(1)(1) - 0 - 2(1)\sqrt{t} \right]$$

$$= 2.$$