Double integrals (over Rectangles)

Exam 3 on Friday: Covers up to Lagrange Multipliers.

Goal: Define integral of a function of 2 variables $\iint_R f(x,y)\,dA$ = a number

in this notation dA represents an "area element"

<u>First definition</u>: Suppose $f(x,y) \geq 0$ then $\iint_R f(x,y)\,dA$ is the volume under the graph of f over the region R

R = some region in the domain of f.

$\underline{\text{Ex}}$ $f(x,y) = e^{-x^2-y^2}$

$R = [-2,2] \times [-2,2]$

$= \{-2 \le x \le 2 \text{ and } -2 \le y \le 2\}$

R is a rectangle

Second definition is by Riemann sums

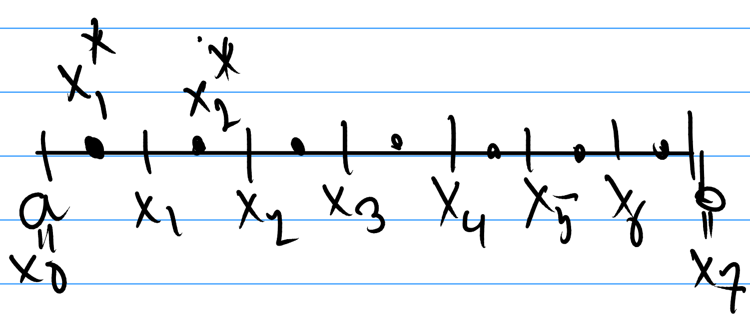
1-var: $f(x)$ interval $[a,b]$

Want $\int_a^b f(x)\,dx$

Riemann $\Rightarrow$ divide interval $[a,b]$ into

n subintervals and sample from each

Assume equal length



$x_i - x_{i-1} = \frac{b-a}{n}$

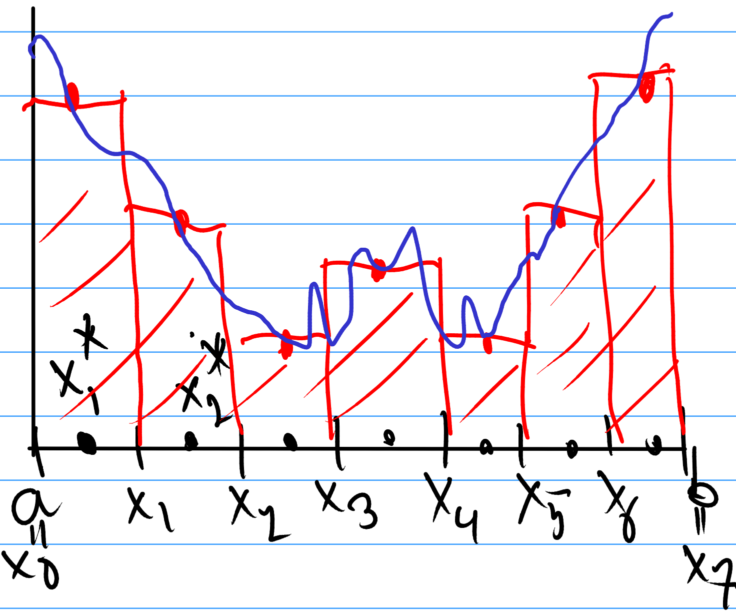
$= \Delta x$

Samples: pick x_i^* in $[x_{i-1}, x_i]$

sample points

Take $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ = Area of the rectangles

This is a Riemann sum



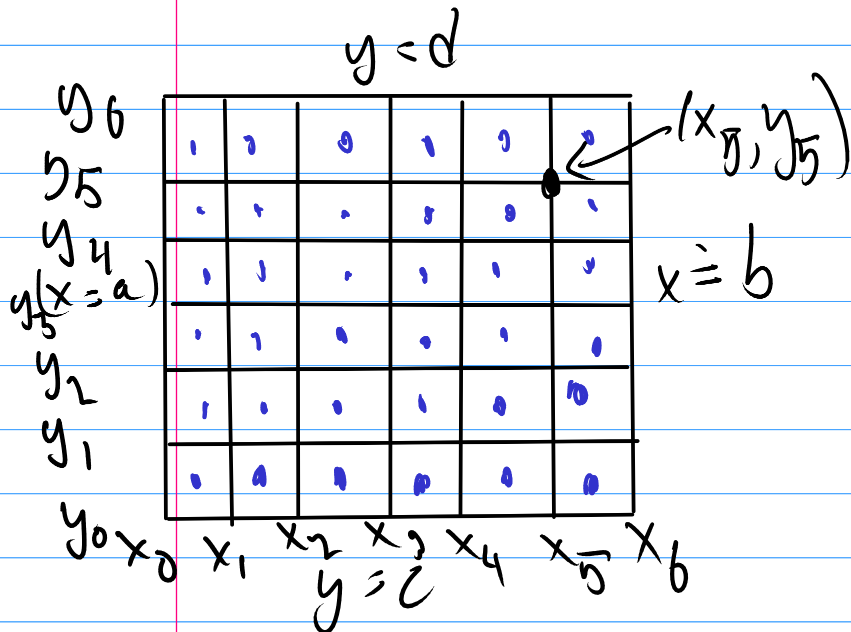
The Riemann sum is an approximation to the true value of the area

The approximation get better if you use a finer subdivision

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{\substack{(\text{finer subdivisions})\\ (n\to\infty)}} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

2 variables $R = [a,b] \times [c,d]$

$\{a \leq x \leq b \text{ and } c \leq y \leq d\}$



Divide rectangle into smaller rectangles of dimensions

$$\Delta x = \frac{b-a}{m}$$

$$\Delta y = \frac{d-c}{n}$$

Now choose sample points:

x_i^* in $[x_{i-1}, x_i]$

y_j^* in $[y_{j-1}, y_j]$

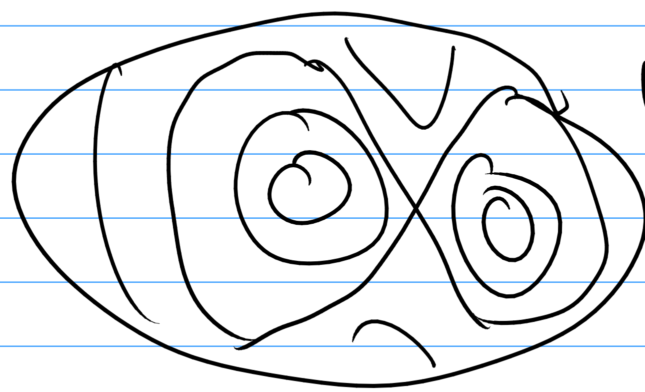
$$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*) \underbrace{\Delta x \, \Delta y}_{\Delta A}$$

ΔA = area of the rectangle in the domain.

This is a two-variable Riemann sum

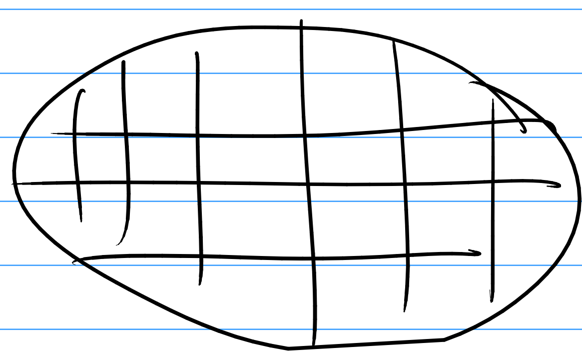
$$\iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta A$$

Example Find the volume of water in lake:



lake

To find the volume of the lake: sample the depth at various points



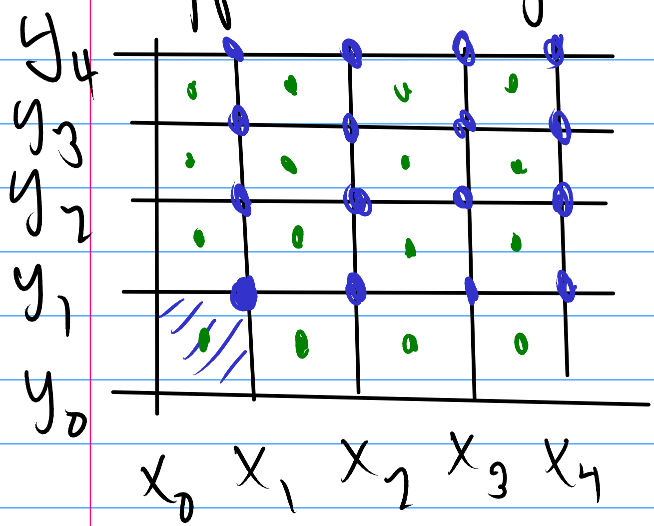
take

$$\sum (\text{depth}) \times \begin{pmatrix}\text{area} \\ \text{of} \\ \text{grid} \\ \text{square}\end{pmatrix}$$

This is a Riemann sum.

Sampling schemes

Upper - Right corner rule



$$(x_i^*, y_j^*) = (x_i, y_j)$$

bottom left

$$(x_i^*, y_j^*) = (x_{i-1}, y_{j-1})$$

Midpoint Rule

$$(x_i^*, y_j^*) = \left(\frac{x_i - x_{i-1}}{2}, \frac{y_j - y_{j-1}}{2} \right)$$

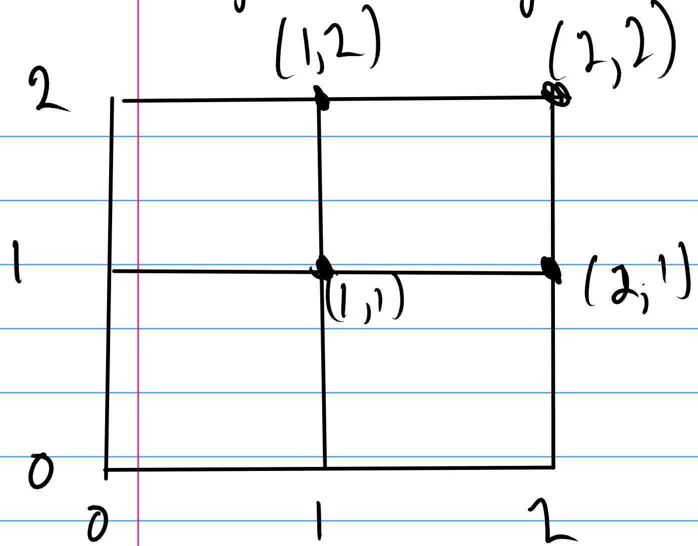
<u>Ex 1</u> $\quad R = \underset{x}{[0,2]} \times \underset{y}{[0,2]}$

$$f(x,y) = 16 - x^2 - 2y^2$$

4 regions, upper right corner rule

Approximate $\iint_R f(x,y)\,dA$

$f(x,y) = 16 - x^2 - 2y^2$



$$\Delta x = \frac{2-0}{2} = 1$$

$$\Delta y = \frac{2-0}{2} = 1$$

$$\Delta A = 1$$

$$RS = f(1,1)\,\Delta A + f(2,1)\,\Delta A + f(1,2)\,\Delta A + f(2,2)\,\Delta A$$

$$= 13 \cdot 1 + 10 \cdot 1 + 7 \cdot 1 + 4 \cdot 1$$

$$= 34$$

Average value:

1-var $\quad f(x)$ $\quad$ $[a,b]$ $\qquad \text{Average}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$

2-var $\quad f(x,y)$

$R = [a,b] \times [c,d]$

$$\text{Average}(f) = \frac{1}{\text{Area}(R)} \iint_R f\,dA = \frac{1}{(b-a)(d-c)} \iint_R f\,dA$$

Linearity:

$$\iint_R [f+g]\,dA = \iint_R f\,dA + \iint_R g\,dA$$

$$\iint_R c f\,dA = c\iint_R f\,dA \quad \text{const. } c$$

Monotonicity: $f \leq g$

$$\iint_R f\,dA \leq \iint g\,dA$$

(obvious if you think about volumes)