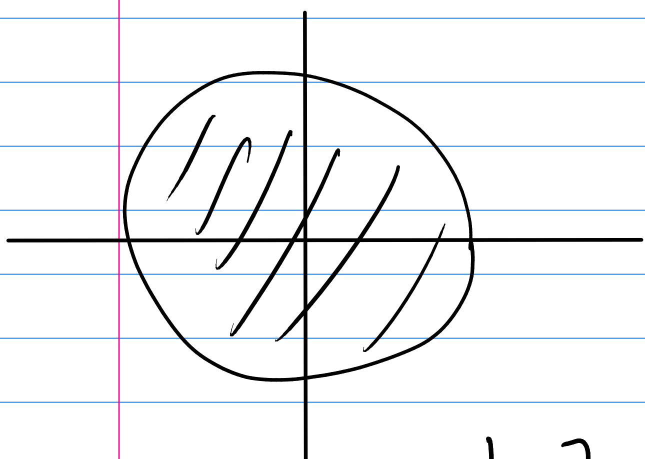
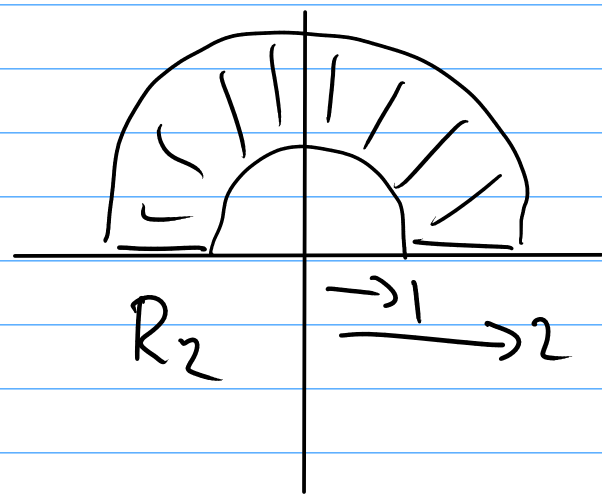
Double integrals in polar coordinates



$R_1 = \{(x,y) \mid x^2+y^2 \le 1\}$



between circle $x^2+y^2=1$
and circle $x^2+y^2=4$
and above the x-axis

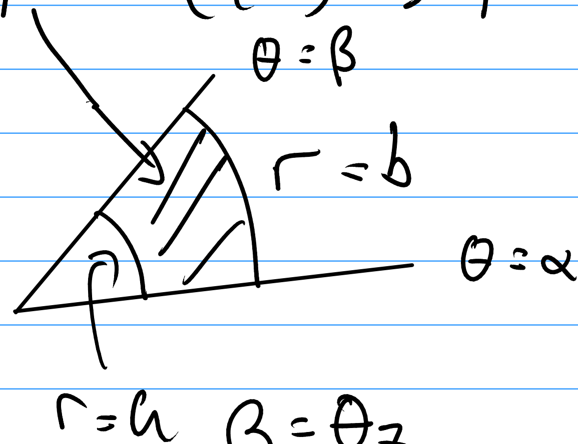
Easier to describe in polar coords.

$$R_1 = \{(r,\theta) \mid r \le 1\}$$

$$R_2 = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

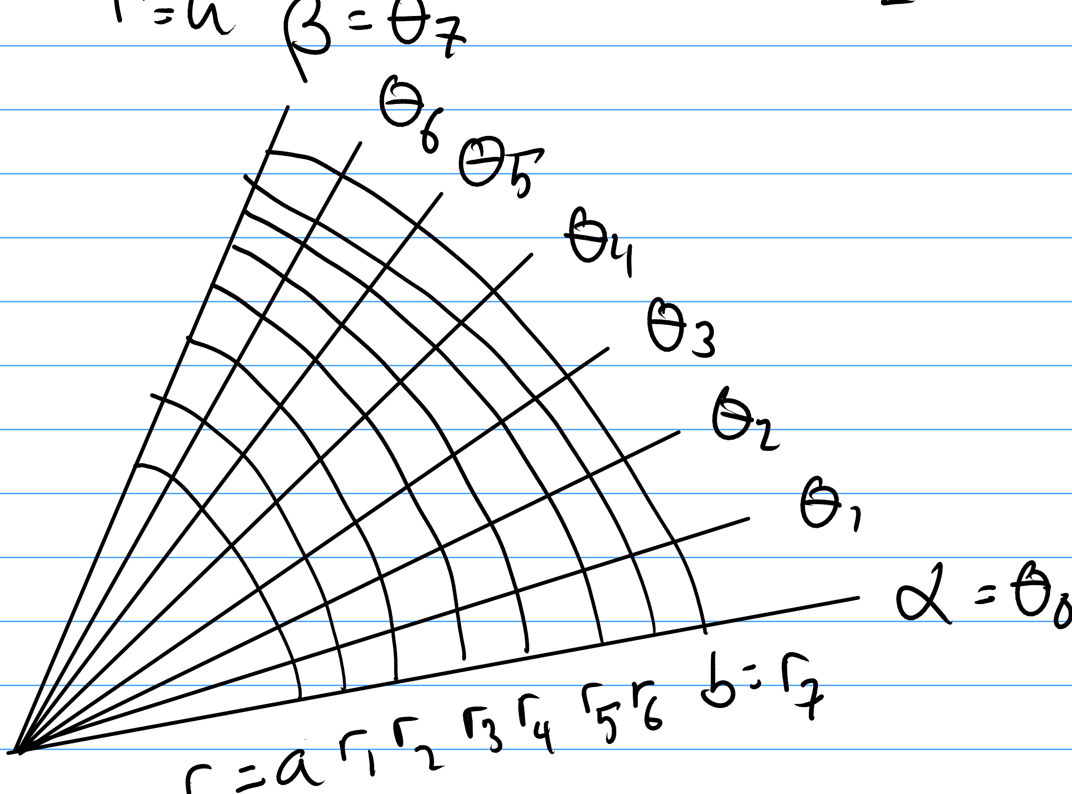
Polar Rectangle

$$R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$



$$A = \frac{1}{2} b^2 (\beta - \alpha) - \frac{1}{2} a^2 (\beta - \alpha)$$

$$= \frac{1}{2}\left[b^2 - a^2\right](\beta - \alpha)$$



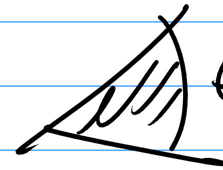
$r : [a, b]$ subdivide into $[r_{i-1}, r_i]$

$$\Delta r = r_i - r_{i-1} = \frac{b-a}{m}$$

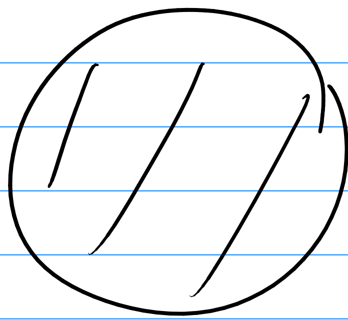
$\theta : [\alpha, \beta]$ subdivide into $[\theta_{j-1}, \theta_j]$ $\Delta\theta = \frac{\beta - \alpha}{n}$

What is the area of the "rectangle"

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

Recall Area ( θ) $= \frac{1}{2} r^2 \theta$

r

Area  $= \pi r^2$

Fraction of circle

$$\frac{\theta}{2\pi}$$

$$\pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$

$$A(R_{ij}) = \frac{1}{2}\left[r_i^2 - r_{i-1}^2\right]\underbrace{(\theta_j - \theta_{j-1})}_{\Delta\theta}$$

$$= \frac{1}{2}(r_i + r_{i-1})\underbrace{(r_i - r_{i-1})}_{\Delta r}\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})\,\Delta r\,\Delta\theta$$

"Midpoint" of R_{ij} : $r_i^* = \frac{1}{2}(r_i + r_{i-1})$

$\theta_j^* = \frac{1}{2}(\theta_j + \theta_{j-1})$

$$A(R_{ij}) = r_i^* \Delta r \Delta \theta = \Delta A_i$$

$\iint_R f(x,y)\,dA$ we can write a Riemann sum using polar rectangles

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(\underset{x_{ij}^*}{\underset{\shortparallel}{r_i^* \cos\theta_j^*}}, \underset{y_{ij}^*}{\underset{\shortparallel}{r_i^* \sin\theta_j^*}}) \Delta A_i$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, \underline{r_i^*}\, \Delta r \Delta \theta$$

This is a riemann sum for

$\iint g(r,\theta)\,dr\,d\theta$ where

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) \cdot r$$

So, if we take the limit $m \to \infty$, $n \to \infty$

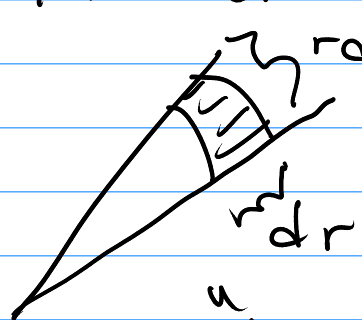
of the Riemann sum, we get

$$\int_\alpha^\beta \int_a^b f(\underbrace{r\cos\theta}, \underbrace{r\sin\theta}) \boxed{r}\, dr\, d\theta$$

change of variables

NOTICE EXTRA FACTOR of r.

17th century justification:



dA = area of "infinitesimal" polar rectangle

"length" = dr

"width" = length of arc subtended by infinitesimal angle $d\theta$

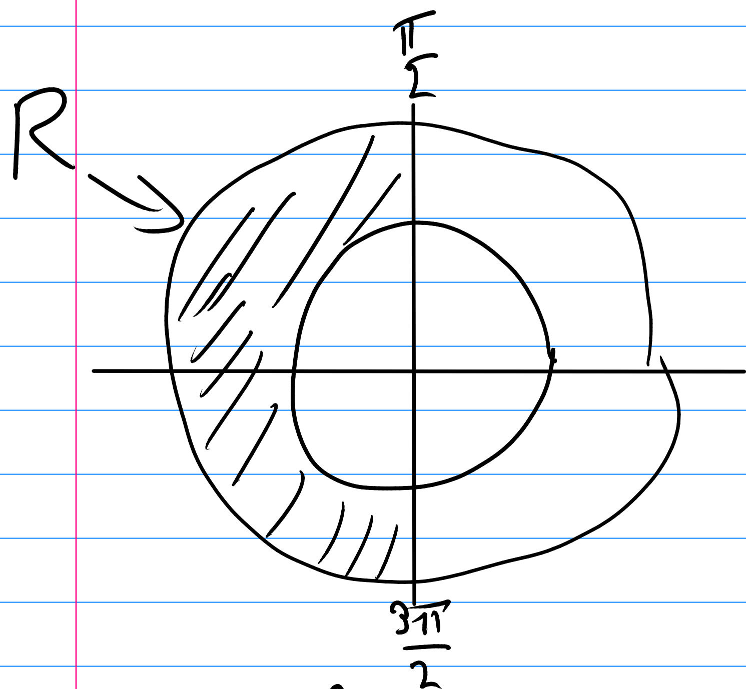
$= r\,d\theta$

$$dA = (r\,d\theta)(dr) = r\,dr\,d\theta$$

Ex $\iint_R (x+y)\,dA$ $R =$ left of y-axis between the circles

$x^2+y^2=1$

$x^2+y^2=4$



$1 \le r \le 2$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 \left[r\cos\theta + r\sin\theta \right] r\,dr\,d\theta$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 \left[r^2 (\cos\theta + \sin\theta) \right] dr\,d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{1}{3} r^3 (\cos\theta + \sin\theta) \right]_{r=1}^{r=2} d\theta$$

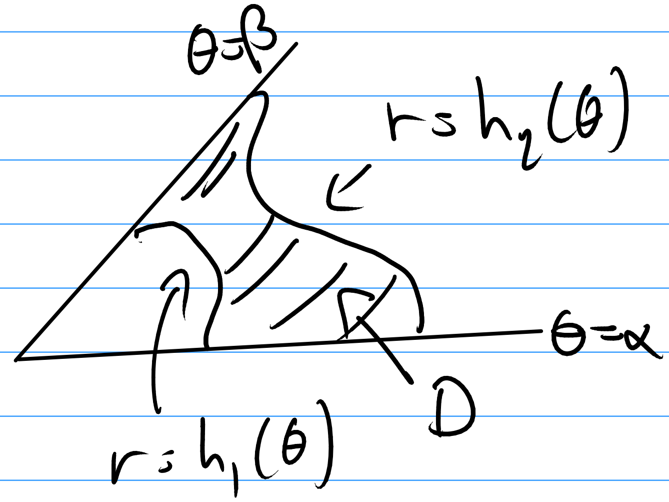
$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{7}{3} (\cos\theta + \sin\theta)\, d\theta$$

$$= \frac{7}{3} \Big[\sin\theta - \cos\theta \Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{7}{3} \Big[-1 - 0 - (1 - 0) \Big] = \frac{-14}{3}$$

Area between two polar equations

$r = h_1(\theta)$, $r = h_2(\theta)$



$$\iint_D f(x,y)\, dA$$

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$

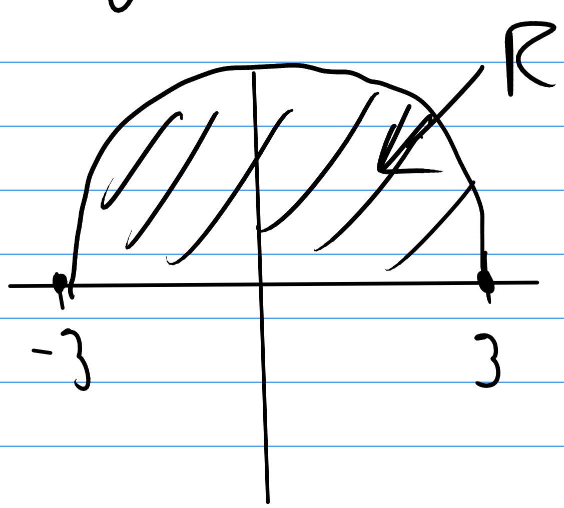
what is $\vec{R}$?

Convert to polar coordinates to make the integral easier.

$x^2+y^2=r^2 \qquad \sin(x^2+y^2)=\sin(r^2)$

$dy\,dx = dA = r\,dr\,d\theta$

$$\int_0^{\pi}\int_0^{3} \sin(r^2)\, r\,dr\,d\theta$$



$y=\sqrt{9-x^2}$ $\qquad x^2+y^2=9$

circle of radius 3

$$\int_0^{\pi} \int_0^{3} \sin(r^2)\, r\, dr\, d\theta$$

$u = r^2 \quad du = 2r\, dr$

$$\int_0^{\pi} \int_0^{9} \frac{1}{2} \sin(u)\, du\, d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{2}(-\cos u) \right]_0^9 d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(1 - \cos 9)\, d\theta = \frac{\pi}{2}(1 - \cos 9)$$