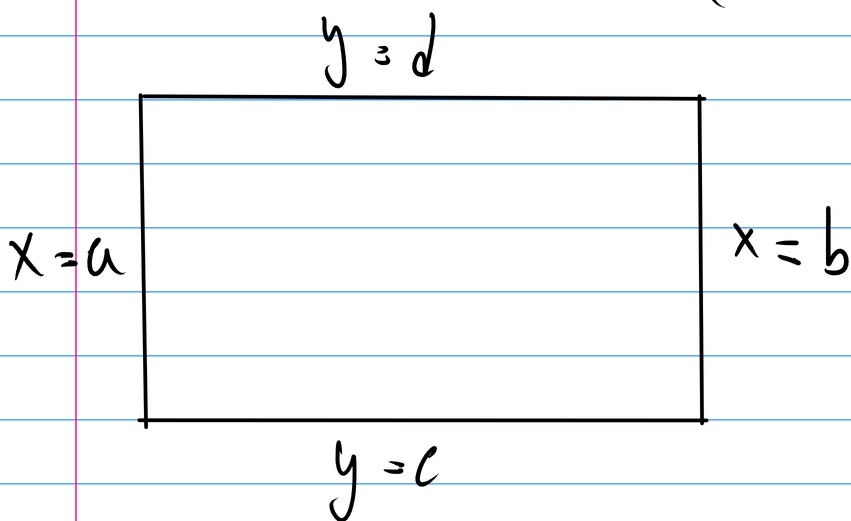
Double Integrals over general Regions

$$\iint_R f(x,y)\,dA \qquad R=[a,b]\times[c,d]$$

$$= \{(x,y) \mid a\le x\le b,\ c\le y\le d\}$$

such that

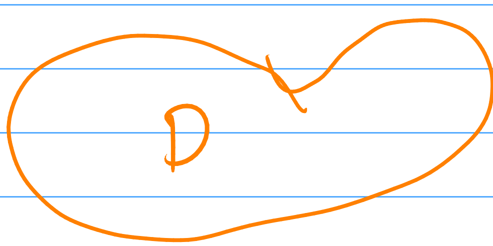


$$\int_a^b\int_c^d f(x,y)\,dy\,dx \quad \text{iterated integral}$$

TODAY want to define / learn how to compute

$$\iint_D f(x,y)\,dA$$ where D is not-necessarily a rectangle

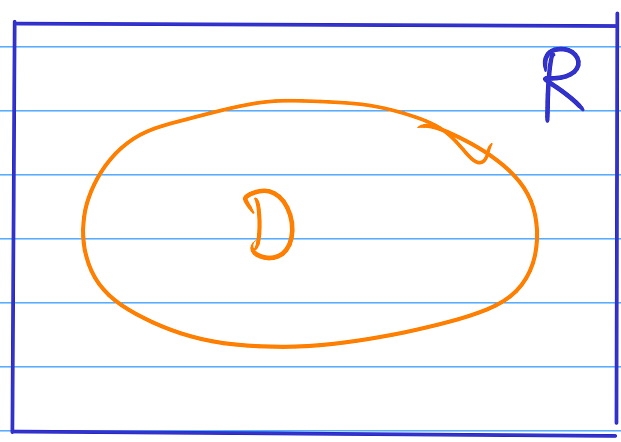
= volume under the graph, lying over D.



Straight-forward but useless method

Extend $f(x,y)$ by zero outside of D, and integrate over a rectangle enclosing D.

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in } D \\ 0 & \text{otherwise} \end{cases}$$



$$\iint_D f(x,y)\,dA$$

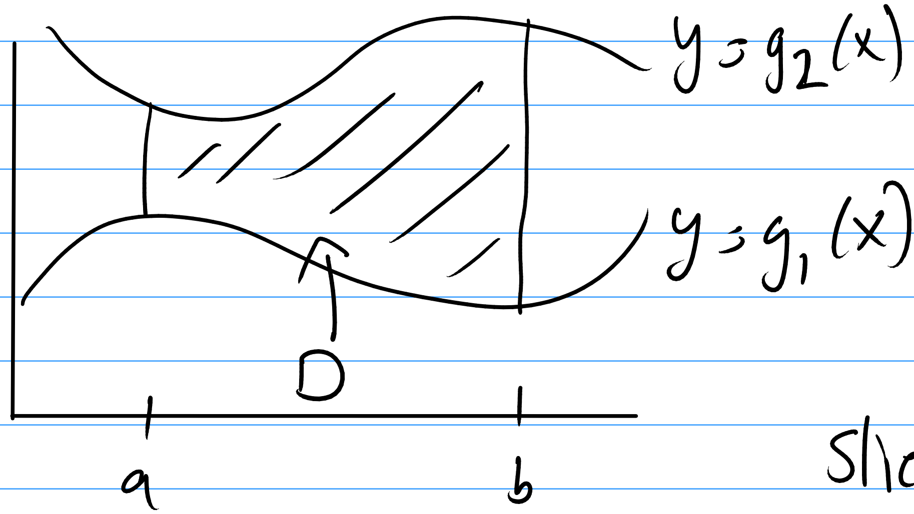
$$= \iint_R F(x,y)\,dA$$

You would use this if you wanted to do a Riemann sum.

Iterated integrals give a better method but it only works well under certain conditions

Type I region: D lies between the graphs of two continuous functions of x

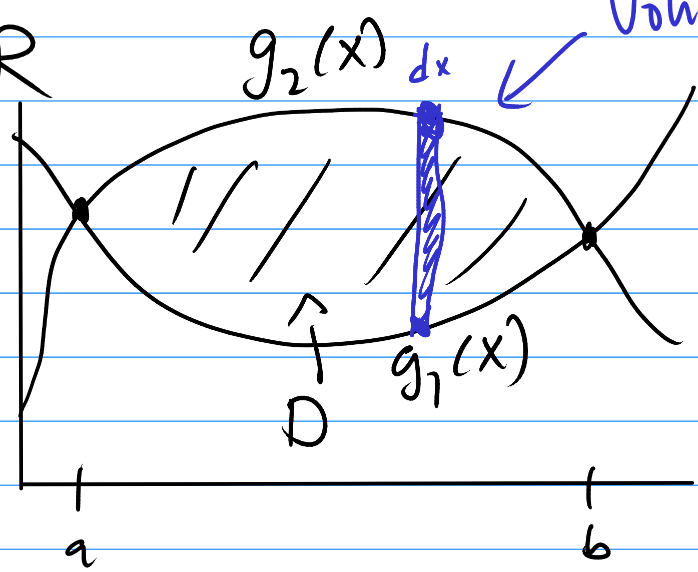
$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$



$$\iint_D f(x,y)\,dA$$

Slice the region vertically

OR



Volume of this slice = Area(cross section) $\cdot\, dx$

$$\text{Area(cross section)} = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

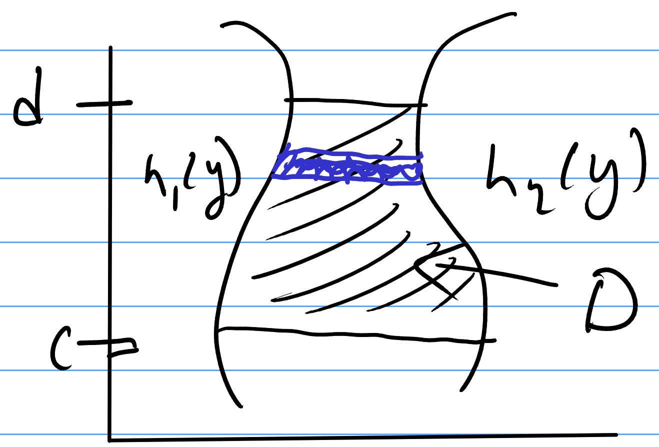
$$\text{Total} = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

For a type I region D

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II region: D lies between the graphs of two continuous functions of y

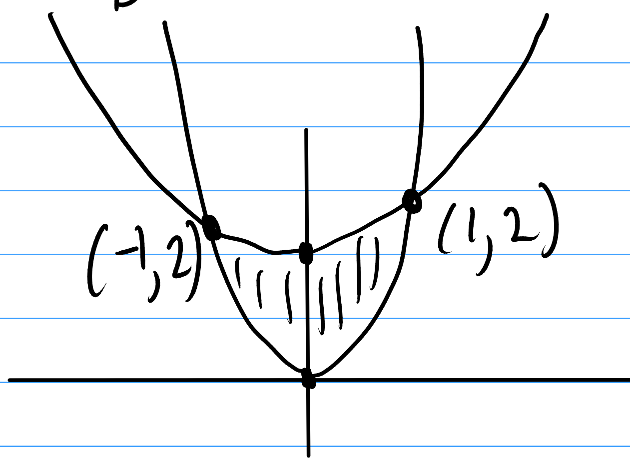
$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$



slicing horizontally.

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

$\underline{\text{Ex}}$ $\iint_D (x+2y)\,dA$ D is bounded by $y=2x^2$, $y=1+x^2$



$2x^2 = y = 1+x^2$

$x^2 = 1$

$x = \pm 1$

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

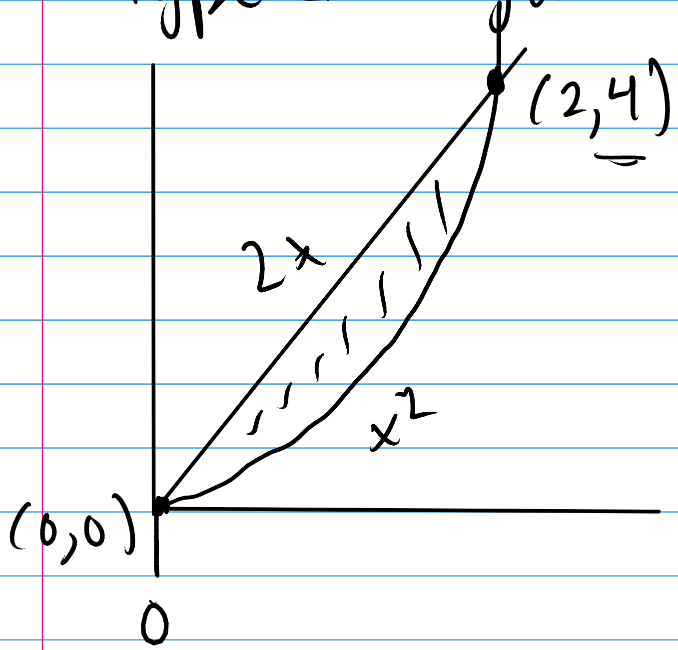
$$= \int_{-1}^{1} \Big[xy + y^2 \Big]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \Big[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \Big] dx$$

$f(x,y)$

Ex $D = \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$

Type I region:



$x = 2$ $\quad x^2 = 4$

$2x = 4$

$$\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$$

This is also a type II region.

$$\left.\begin{aligned} x^2 \le y &\iff x \le \sqrt{y} \\ y \le 2x &\iff \frac{y}{2} \le x \end{aligned}\right\} \quad \frac{y}{2} \le x \le \sqrt{y}$$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

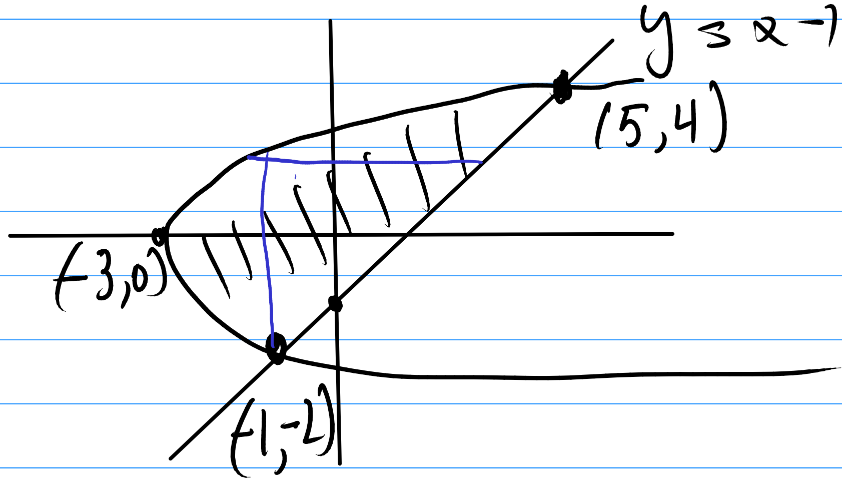
D is

Ex $\iint_D xy\, dA$ between $y = x-1 \to x = y+1$

$y^2 = 2x+6$

$\downarrow$

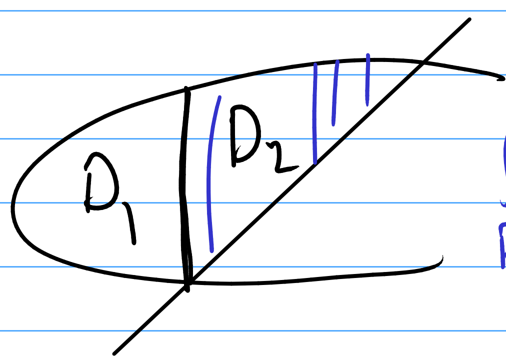
$x = \dfrac{y^2-6}{2}$



Type II slice horizontally:

$$\int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} xy\, dx\, dy$$

Another way split it into two type I integrals

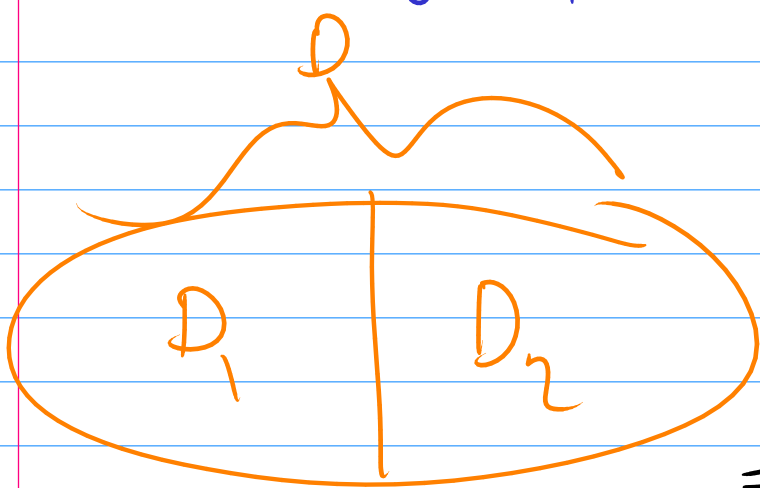


slice vertically

$$\iint_{D_1} xy\, dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\, dy\, dx$$

$$\iint_{D_2} xy\,dA = \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

Add these two together



$$\iint_D f(x,y)\,dA$$

$$= \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$