The Dot product

This is a way of "multiplying" two vectors to get a number. It is easy to express in components:

The Dot Product of vectors
$\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

is $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, which is a number.

also called scalar product or inner product.

in 2d: $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

Ex $\langle 3,5\rangle \cdot \langle 0,-1\rangle$

$$= 3\cdot 0 + 5\cdot(-1) = -5$$

$$(\vec{i}+2\vec{j}-\vec{k})\cdot(2\vec{j}-\vec{k})$$

$$= \langle 1,2,-1\rangle \cdot \langle 0,2,-1\rangle$$

$$= 1\cdot 0 + 2\cdot 2 + (-1)\cdot(-1)$$

$$= 0 + 4 + 1 = 5$$

Properties of dot product:

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative law)
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive law)
4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$
5. $\vec{0}\cdot\vec{a} = 0$

Property 1 is interesting

Proof of 1: $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3$$
$$= a_1^2 + a_2^2 + a_3^2$$
$$= |\vec{a}|^2$$

Dot products of basis vectors:

$$\vec{i} \cdot \vec{i} = \langle 1,0,0 \rangle \cdot \langle 1,0,0 \rangle = 1$$
$$\vec{i} \cdot \vec{j} = \langle 1,0,0 \rangle \cdot \langle 0,1,0 \rangle = 0$$

similarly

$\vec{i} \cdot \vec{i} = 1$ $\quad \vec{j} \cdot \vec{j} = 1$ $\quad \vec{k} \cdot \vec{k} = 1$

$\vec{i} \cdot \vec{j} = 0 = \vec{j} \cdot \vec{i}$

$\vec{i} \cdot \vec{k} = 0 = \vec{k} \cdot \vec{i}$

$\vec{j} \cdot \vec{k} = 0 = \vec{k} \cdot \vec{j}$

Using this and the distributive law, we can compute dot products in i-j-k notation.

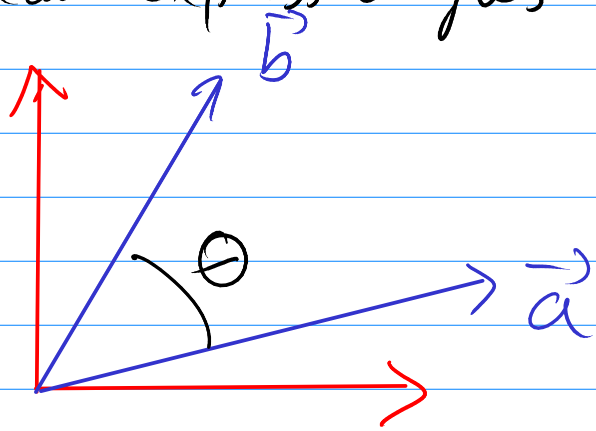
$$
\begin{aligned}
&(1\vec{i}+2\vec{j}+3\vec{k})\cdot(\pi\vec{i}+e\vec{j}+\phi\vec{k}) \\
&= (1\cdot\pi)\,\vec{i}\cdot\vec{i}+(2\cdot e)\,\vec{j}\cdot\vec{j}+(3\cdot\phi)\,\vec{k}\cdot\vec{k} \\
&\quad \left.\begin{aligned}
&+(1\cdot e)\,\vec{i}\cdot\vec{j}+(1\cdot\phi)\,\vec{i}\cdot\vec{k} \\
&+(2\cdot\pi)\,\vec{j}\cdot\vec{i}+(2\cdot\phi)\,\vec{j}\cdot\vec{k} \\
&+(3\cdot\pi)\,\vec{k}\cdot\vec{i}+(3\cdot e)\,\vec{k}\cdot\vec{j}
\end{aligned}\right\}\text{all this is }0\text{!} \\
&= 1\pi+2e+3\phi
\end{aligned}
$$

Some problems are simpler:

$$
\begin{aligned}
&(1\vec{i}+2\vec{j}+3\vec{k})\cdot\vec{k} \\
&= 1\,\vec{i}\cdot\vec{k}+2\,\vec{j}\cdot\vec{k}+3\,\vec{k}\cdot\vec{k} \\
&= 0+0+3=3.
\end{aligned}
$$

Geometric meaning of dot product.

Can express angles between vectors.



Let θ be the angle between the vectors.

Theorem:

$$\vec{a} \bullet \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

Or

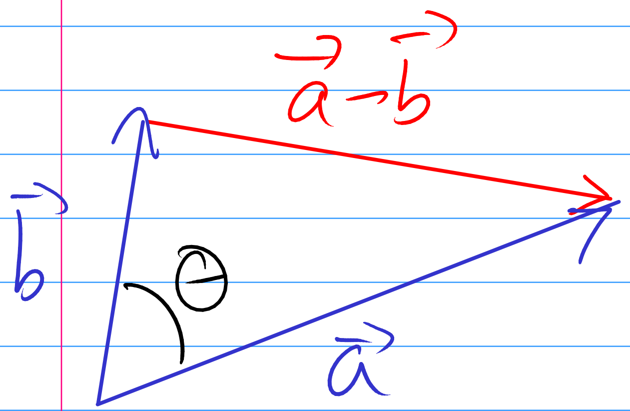
$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

(unless $\vec{a}$ or $\vec{b}$ is zero).

NOTE: we always take the angle θ between the vectors to be BETWEEN 0 AND π.

(0° and 180°)

This theorem follows from the law of cosines:



$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

By dot product rules, we also have

$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$
$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\vec{a}\cdot\vec{b}$$
$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$$

Comparing these equations

$$-2|\vec{a}||\vec{b}|\cos\theta = -2\vec{a}\cdot\vec{b}$$
$$|\vec{a}||\vec{b}|\cos\theta = \vec{a}\cdot\vec{b} \quad \text{QED}$$

Ex If $|\vec{a}| = 6$ and $|\vec{b}| = 5$, and the angle between is $2\pi/3$, then

$$\vec{a} \cdot \vec{b} = 6 \cdot 5 \cdot \cos\frac{2\pi}{3} = 6 \cdot 5 \cdot \frac{1}{2} = 15$$

Ex What is the angle between the vectors $\vec{a} = \langle -8, 6 \rangle$, $\vec{b} = \langle \sqrt{7}, 3 \rangle$

$$|\vec{a}| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

$$|\vec{b}| = \sqrt{7 + 3^2} = \sqrt{16} = 4$$

$$\vec{a} \cdot \vec{b} = (-8)(\sqrt{7}) + 6 \cdot 3 = 18 - 8\sqrt{7}$$

$$= -3.17$$

So $\cos\theta = \dfrac{-3.17}{4 \cdot 10} \approx -0.079$

$$\theta \approx \cos^{-1}(-0.079) \approx 1.65 \text{ rad}$$

$$\approx 94.5^\circ$$

Two vectors are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$ (that is, 90°)

since $\cos\frac{\pi}{2} = \cos 90^\circ = 0$,

We find

$\vec{a}$ and $\vec{b}$ are orthogonal ($\vec{a} \perp \vec{b}$)

if and only if

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$$

More generally, if $\vec{a}\cdot\vec{b}$ is positive

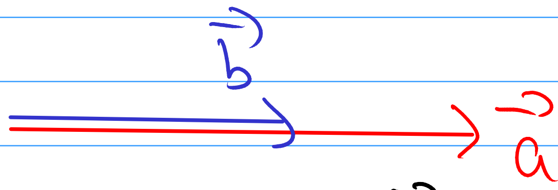
$$\cos\theta > 0, \quad \text{so } 0 < \theta < \frac{\pi}{2}$$

if $\vec{a}\cdot\vec{b}$ is negative,

$$\cos\theta < 0 \quad \text{so } \frac{\pi}{2} < \theta \leq \pi$$

If the vectors $\vec{a}$ and $\vec{b}$ point in the same direction, $\theta = 0$

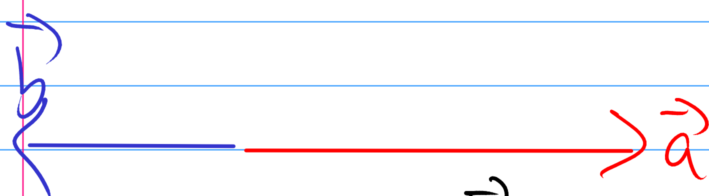
$\cos\theta = 1$



Hence $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

If the vectors $\vec{a}$ and $\vec{b}$ point in opposite directions, $\theta = \pi$

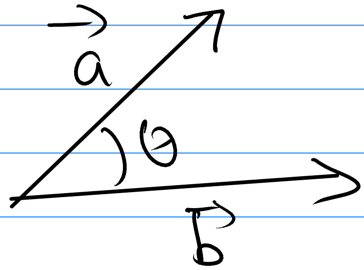
$\cos\pi = -1$



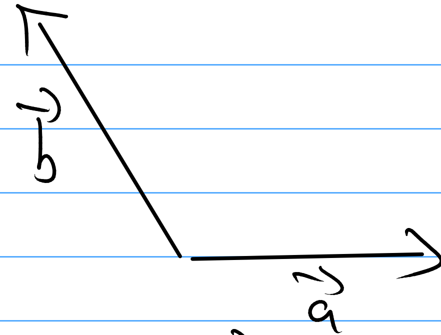
Hence $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

In general, since $|\cos\theta| \leq 1$, we have the Cauchy-Schwarz inequality

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$



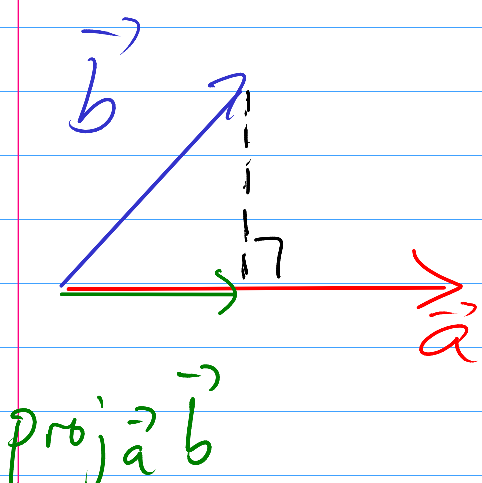
$\vec{a} \cdot \vec{b} > 0$

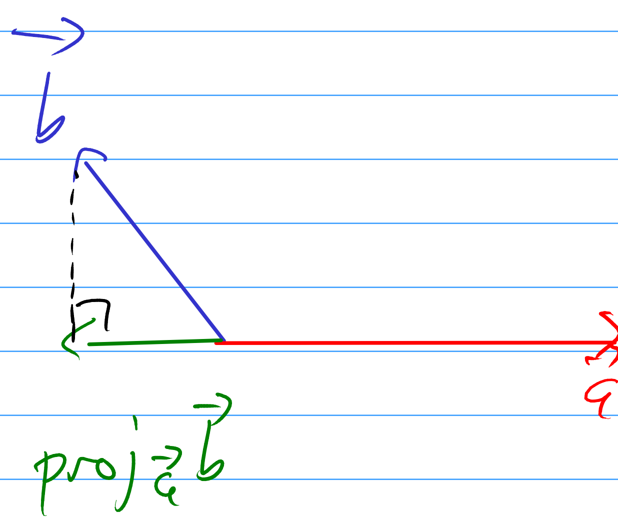


$\vec{a} \cdot \vec{b} < 0$

What emerges is that $\vec{a} \cdot \vec{b}$ measures the extent to which $\vec{a}$ and $\vec{b}$ point in the same, or opposite, directions (in a way).

To make this idea precise, we introduce projections:





The vector projection of $\vec{b}$ onto $\vec{a}$

$\text{proj}_{\vec{a}} \vec{b}$ is the vector parallel to $\vec{a}$ as in the above figure.

Another characterization is that $\text{proj}_{\vec{a}} \vec{b}$ is the vector parallel to $\vec{a}$ such that

$(\vec{b} - \text{proj}_{\vec{a}} \vec{b})$ is perpendicular to $\vec{a}$

Can use this to derive a formula.

$\text{proj}_{\vec{a}} \vec{b} = c\vec{a}$ for some scalar c that we need to determine.

$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$ b/c perpendicular

$$\Rightarrow \vec{b} \cdot \vec{a} - c\,\vec{a} \cdot \vec{a} = 0$$

$$\Rightarrow c = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}$$

Thus $\text{proj}_{\vec{a}}\vec{b} = \left(\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}$

The scalar projection or component of $\vec{b}$ along $\vec{a}$ is the signed magnitude of the vector projection

$$\left|\text{proj}_{\vec{a}}\vec{b}\right| = \left|\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}\right| = \frac{|\vec{b}\cdot\vec{a}|}{|\vec{a}|^2}|\vec{a}|$$

$$= \frac{|\vec{b}\cdot\vec{a}|}{|\vec{a}|}$$

$\text{comp}_{\vec{a}}\vec{b}$ has this absolute value, and the same sign as $\vec{b}\cdot\vec{a}$, so

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$$ (can take this as definition)

To summarize

Scalar projection

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$$

Vector projection

$$\text{proj}_{\vec{a}}\vec{b} = (\text{comp}_{\vec{a}}\vec{b})\frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$$

Also: $\vec{a}\cdot\vec{b} = (\text{comp}_{\vec{a}}\vec{b})|\vec{a}|$

Special case: if $\vec{u}$ is a unit vector

$|\vec{u}| = 1$, these formulas simplify

$$\text{comp}_{\vec{u}}\vec{b} = \vec{b}\cdot\vec{u}$$

$$\text{proj}_{\vec{u}}\vec{b} = (\vec{b}\cdot\vec{u})\vec{u}$$

Ex Find scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

$\vec{a} = \langle 3, 6, -2 \rangle \quad \vec{b} = \langle 1, 2, 3 \rangle$

$$|\vec{a}| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{9+36+4} = \sqrt{49} = 7$$

$$\vec{b} \cdot \vec{a} = 1 \cdot 3 + 2 \cdot 6 + 3 \cdot (-2) = 3 + 12 - 6 = 9$$

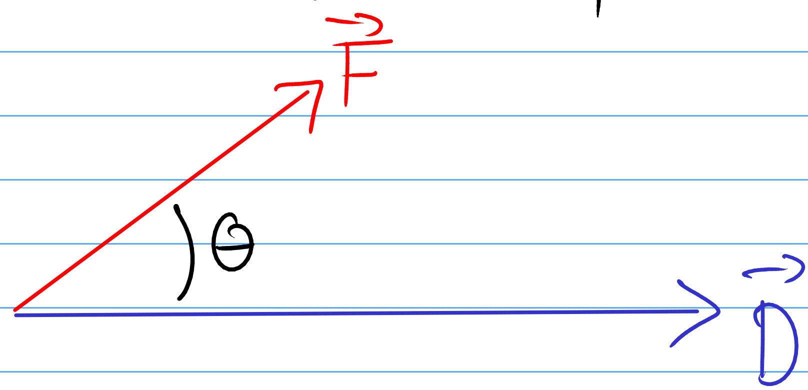
$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \frac{9}{7}$$

$$\text{proj}_{\vec{a}} \vec{b} = \left(\text{comp}_{\vec{a}} \vec{b}\right) \frac{\vec{a}}{|\vec{a}|} = \frac{9}{7} \frac{1}{7} \vec{a}$$

$$= \frac{9}{49} \langle 3, 6, -2 \rangle = \left\langle \frac{27}{49}, \frac{54}{49}, \frac{-18}{49} \right\rangle$$

Physics application of dot product: WORK

Work = Force × Displacement



Force and displacement are actually vectors, and Work is the dot product

$$W = \vec{F} \cdot \vec{D} = \left(|\vec{F}| \cos\theta\right)|\vec{D}|$$

$$= \left(\text{comp}_{\vec{D}}\vec{F}\right)|\vec{D}|$$

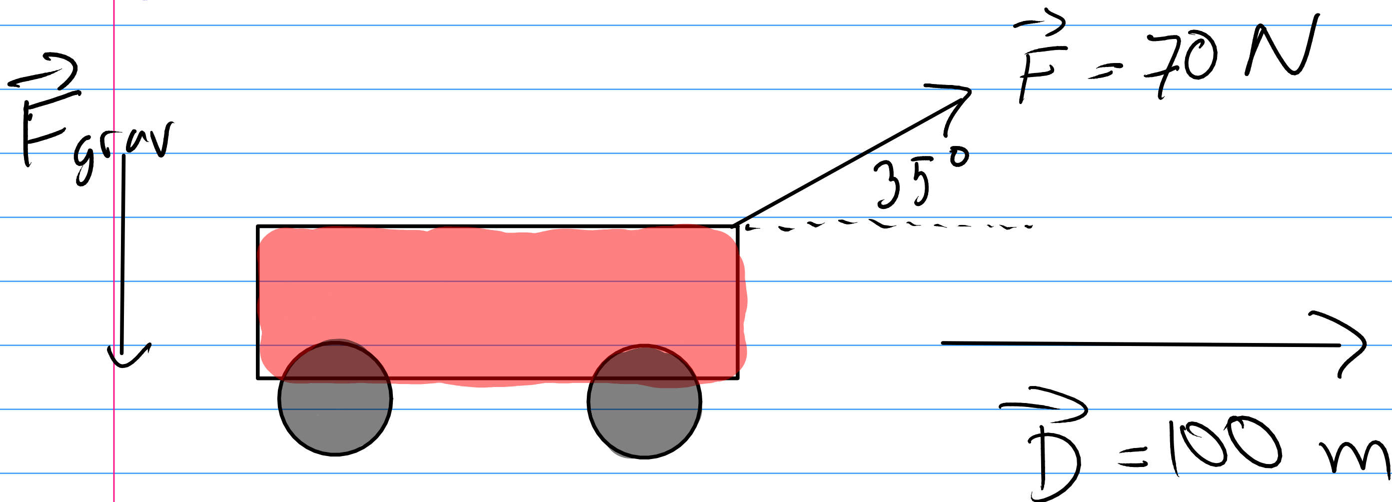
[Work is the net change in total energy (Kinetic + Potential) over the course of the process]

Ex a wagon is pulled 100 meters by a force of 70 Newtons, with the handle held at 35° from horizontal

How much Work is done by

(a) The pulling force

extra [(b) The force of gravity?]



$$W = \vec{F} \cdot \vec{D} = (70\text{ N})(100\text{ m})\cos 35°$$

$$\approx 5734 \text{ N} \cdot \text{m} = 5734 \text{ J (joules)}$$

[Note: we didn't do this page in lecture, so consider "EXTRA"]

For part (b), we recall that

$$\left|\vec{F}_{grav}\right| = mg \qquad m = \text{mass (in kg)}$$
$$g = 9.8\ m/s^2$$

Oh no! we need the mass of the wagon, don't we?

Actually, in this case it doesn't matter, because $\vec{F}_{grav}$ is perpendicular to the direction of motion, and hence to $\vec{D}$.

$$W_{grav} = \vec{F}_{grav} \cdot \vec{D} = mg\,|\vec{D}|\cos\frac{\pi}{2}$$

$$= 0$$