Cross Products

The cross product or vector product of two 3-dimensional vectors

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$
$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

is given in components by

$$\vec{a} \times \vec{b} =$$
$$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

This seems very weird at first, but we'll see that it actually has a lot of geometric meaning.

Note: the cross product is special to 3 dimensions. It doesn't exist in 2d!

The formula itself is kind of hard to remember, so we rewrite it in terms of determinants:

The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Eg

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2 \cdot 4 - 1 \cdot (-6) = 14$$

The determinant of a 3x3 matrix can be expressed in terms of 2x2 determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Eg: $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix}$

$$= 1 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0-4) - 2(6+5) + (-1)(12-0)$$

$$= -38$$

Using determinants

$\vec{a} \times \vec{b} =$

$= \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

$= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$

$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

First row = $\vec{i}\ \vec{j}\ \vec{k}$

second row $\vec{a} = \langle a_1, a_2, a_3 \rangle$

third row $\vec{b} = \langle b_1, b_2, b_3 \rangle$

this packages $\vec{a} \times \vec{b}$ as a determinant.

(It's a bit of a mnemonic device, since $\vec{i}, \vec{j}, \vec{k}$ are vectors not numbers)

Ex $\vec{a} = \langle 1, 3, 4 \rangle$ $\vec{b} = \langle 2, 7, -5 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$

$$= \vec{i}(-15 - 28) - \vec{j}(-5 - 8) + \vec{k}(7 - 6)$$

$$= -43\vec{i} + 13\vec{j} + \vec{k} = \langle -43, 13, 1 \rangle$$

Apology: If these definitions seem unmotivated, that's because they are. As far as I know, there isn't a simple convincing motivation for the determinant and the cross product, but they do turn out to be very useful.

Geometric uses of Cross Products

Use 1: Manufacturing orthogonal vectors.

<u>Theorem:</u> The cross product $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$:

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0.$$

This is a straightforward but lengthy computation:

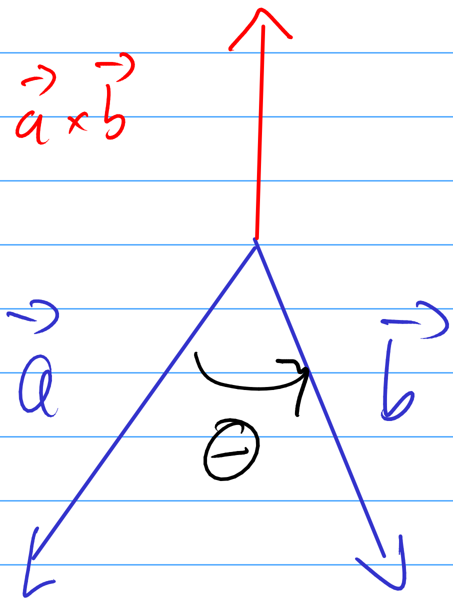
$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= (a_2 b_3 - a_3 b_2) a_1 - (a_1 b_3 - a_3 b_1) a_2 + (a_1 b_2 - a_2 b_1) a_3$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_1 a_2 b_3 + a_2 a_3 b_1 + a_1 a_3 b_2 - a_2 a_3 b_1$$

$$= 0$$

The direction of $\vec{a} \times \vec{b}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, and furthermore is determined by the right hand rule



θ the angle between $(0 \leq \theta \leq \pi)$

$\vec{a}$ = index finger
$\vec{b}$ = middle finger
$\vec{a} \times \vec{b}$ = thumb

(Or, curl your fingers from $\vec{a}$ to $\vec{b}$)

Make sure to use your right hand (you may need to put down your pencil).

Note that the order matters:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

Use 2: relationship to angle θ and areas of parallelograms:

The magnitude of the cross product satisfies

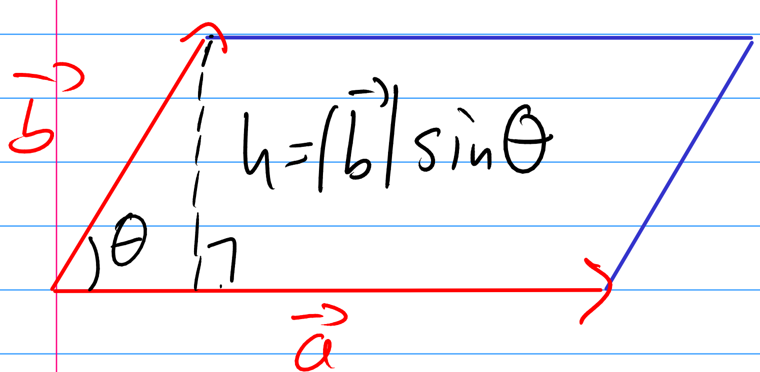
$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

(really need $0 \leq \theta \leq \pi$ here).

This also boils down to a lengthy computation, which we omit.

This means that

$|\vec{a} \times \vec{b}|$ is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$



Area = base × height

$= |\vec{a}|\, h = |\vec{a}|\,|\vec{b}| \sin\theta$

Example of uses 1 & 2:

Find a vector perpendicular to the plane containing the points

$P(1,4,6) \quad Q(-2,5,-1) \quad R(1,-1,1)$

First step: the vectors $\vec{PQ}$ and $\vec{PR}$ lie <u>in</u> the plane:

$$\vec{PQ} = (-2-1)\vec{i} + (5-4)\vec{j} + (-1-6)\vec{k}$$
$$= -3\vec{i} + 1\vec{j} - 7\vec{k}$$

$$\vec{PR} = (1-1)\vec{i} + (-1-4)\vec{j} + (1-6)\vec{k}$$
$$= 0\vec{i} - 5\vec{j} - 5\vec{k}$$

Second Step: we want something $\perp$ to $\vec{PQ}$ and $\vec{PR}$, so use cross product.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

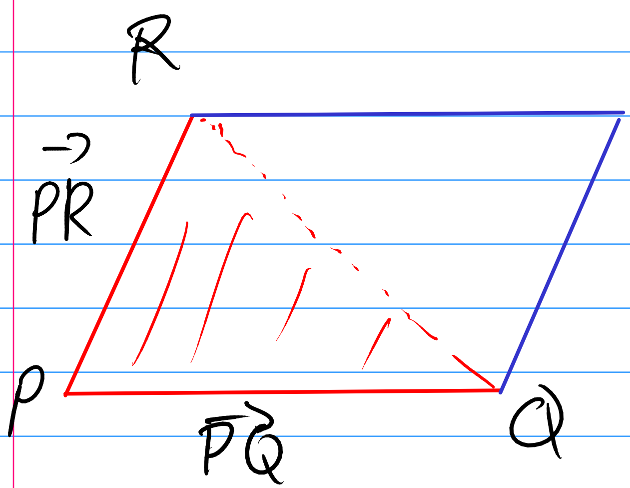
$$= \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix} \vec{k}$$

$$= (-5-35)\vec{i} - (15)\vec{j} + (15)\vec{k}$$

$$= -40\vec{i} - 15\vec{j} + 15\vec{k}.$$

So $\langle -40, -15, 15 \rangle$ is $\perp$ to the plane containing P, Q, and R.

Second question: Area of triangle w/ P, Q, R as vertices



Area(Parallelogram) $= |\vec{PQ} \times \vec{PR}|$

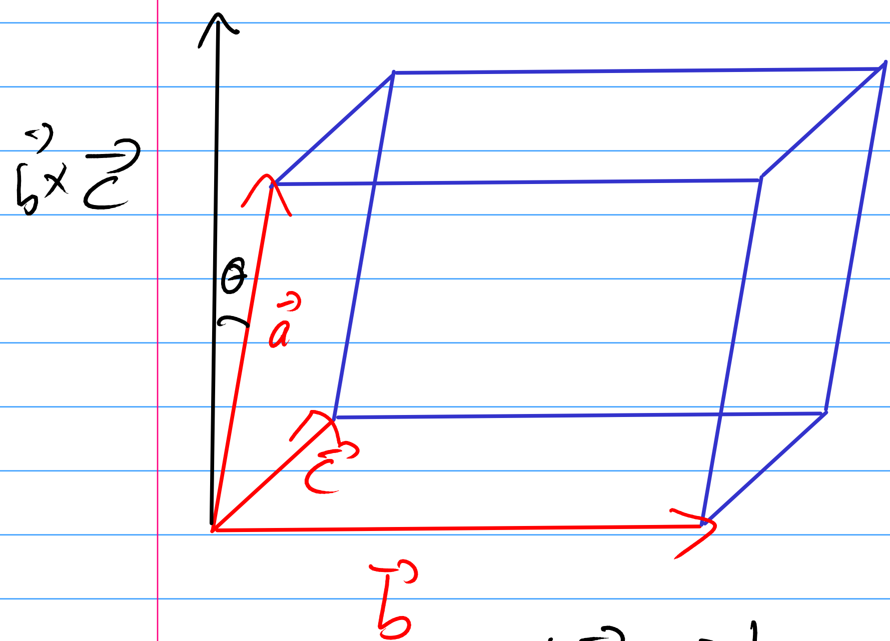
Area(triangle) $= \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$$\text{Area(triangle)} = \frac{1}{2} \left| \langle -40, -15, 15 \rangle \right|$$

$$= \frac{1}{2} \sqrt{40^2 + 15^2 + 15^2} = \frac{1}{2} 5\sqrt{82} = \frac{5}{2}\sqrt{82}$$

Use 3 of cross product:

volume of 3 dimensional parallelepiped



Volume

$= |\vec{a}\cdot(\vec{b}\times\vec{c})|$

Reason: $|\vec{b}\times\vec{c}|$ is area of base,

$|\vec{a}|\cos\theta$ is height

Volume = (area of base) $\times$ (height).

The expression $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product.

Use 3.5: coplanarity

3 vectors $\vec{a}, \vec{b}, \vec{c}$ are coplaner

if they lie in a single plane.

In this case the volume of the parallelepiped is 0.

Ex Show that

$\vec{a} = \langle 1, 4, -7 \rangle$

$\vec{b} = \langle 2, -1, 4 \rangle$ are coplanar

$\vec{c} = \langle 0, -9, 18 \rangle$

Need to show $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Well $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

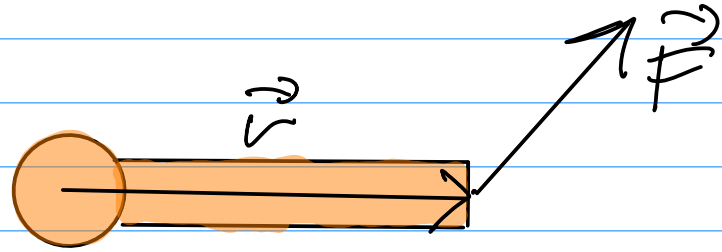
$$= \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(18) - 4(36) + (-7)(-18)$$

$$= 0.$$

Use 4 of cross products:

physical quantity of torque:

(twisting force)



$\vec{r}$ = vector from center of rotation to point where force is applied

$\vec{F}$ = Force vector

Torque $\vec{\tau} = \vec{r} \times \vec{F}$

Direction is along axis of rotation:

(in this case out of screen)

Magnitude = strength of twisting

Properties of cross products

Basis vectors

$\vec{i} \times \vec{j} = \vec{k}$ $\quad\quad$ $\vec{j} \times \vec{i} = -\vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$ $\quad\quad$ $\vec{k} \times \vec{j} = -\vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$ $\quad\quad$ $\vec{i} \times \vec{k} = -\vec{j}$

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} + \vec{k} = 0$

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$