Comparison tests

So far, we have developed a few tests for convergence / divergence.

* Test for divergence:

If $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges

* Integral test:

If $f(x)$ is a positive continuous decreasing function for $x > M$, then

$$\sum_{i=M}^{\infty} a_i \begin{cases} \text{converges if } \int_M^\infty f(x)\,dx \text{ converges} \\ \text{diverges if } \int_M^\infty f(x)\,dx \text{ diverges} \end{cases}$$

* p-test (special case of integral test)

$$\sum_{i=1}^{\infty} \frac{1}{i^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

* Geometric series $\sum_{i=1}^{\infty} ar^{i-1}$ converges if $|r| < 1$
diverges if $|r| \geq 1$

Comparison tests let us use knowledge about one series' convergence or divergence to treat other series with a similar form.

<u>Comparison test</u>: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of POSITIVE numbers.

1) If $a_n \leq b_n$, and $\sum_{n=1}^{\infty} b_n$ converges,
THEN $\sum_{n=1}^{\infty} a_n$ converges as well

2) If $b_n \leq a_n$, and $\sum_{n=1}^{\infty} b_n$ diverges,
THEN $\sum_{n=1}^{\infty} a_n$ diverges as well.

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$. As $n \to \infty$ behavior of terms is dominated by the n in the numerator and the n^3 in the denominator.

In fact:
$$\frac{n}{2n^3+1} = \frac{1}{2n^2+\frac{1}{n}} < \frac{1}{2n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-test),
The original series <u>converges</u>.

In fact, we only need the inequality $a_n \le b_n$ or $b_n \le a_n$ to hold for all but finitely many of the terms:

Ex $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Observe: $\frac{\ln 1}{1} = 0 < \frac{1}{1}$

$\frac{\ln 2}{2} < \frac{1}{2}$

BUT $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and the terms $\frac{\ln n}{n}$ are eventually bigger than those of the harmonic series, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges as well.

what about $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

well, $\frac{1}{2^n-1} > \frac{1}{2^n}$. But $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges,

so this inequality is not usefull.

Yet, $\frac{1}{2^n-1}$ and $\frac{1}{2^n}$ are almost the same size,

so convergence of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ should imply convergence

of $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ More precisely:

<u>Theorem</u> (Limit comparison test)

Suppose $\sum a_n$ and $\sum b_n$ are series with <u>positive</u> terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, and c is <u>finite</u> and <u>$c > 0$</u>,

then either <u>both</u> <u>series converge</u> or <u>both</u> <u>diverge</u>.

<u>Intuition:</u> we have $a_n \approx c b_n$ for large n, (say $n \geq N$)

so $\sum_{n=N}^{\infty} a_n \approx c \sum_{n=N}^{\infty} b_n$.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ use $a_n = \frac{1}{2^n-1}$, $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0$$

The limit exists, and since $\sum b_n = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

converges (it's a geometric series),

we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is <u>convergent</u> as well.

6

$\underline{\text{Ex}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

General idea: keep only highest powers of n and use limit comparison test

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}} \quad \text{vs.} \quad \frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^{5/2}} = \frac{1}{n^{1/2}} = b_n$$

$$\frac{a_n}{b_n} = \frac{2n^2+3n}{\sqrt{5+n^5}} \Big/ \frac{1}{n^{1/2}} = \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}}$$

$$= \frac{2+\frac{3}{n}}{\sqrt{\frac{5}{n^5}+1}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{\sqrt{\frac{5}{n^5}+1}} = \frac{2+0}{\sqrt{0+1}} = 2$$

The limit is finite and positive, so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges (p-test), $\displaystyle\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$

diverges as well.

Ex $\sum_{n=1}^{\infty} \frac{n+1}{n4^n}$

One way, use limit comparison with $\sum_{n=1}^{\infty} \frac{1}{4^n}$, a convergence geometric series.

$$\lim_{n\to\infty} \frac{n+1}{n4^n} \Big/ \frac{1}{4^n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

so $\sum_{n=1}^{\infty} \frac{n+1}{n4^n}$ is convergent as well.

Another way, $= \sum_{n=1}^{\infty} \frac{n}{n4^n} + \sum_{n=1}^{\infty} \frac{1}{n4^n}$

* $\sum_{n=1}^{\infty} \frac{n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{4^n}$ convergent geometric series.

* $\sum_{n=1}^{\infty} \frac{1}{n4^n}$; since, $\frac{1}{n4^n} < \frac{1}{4^n}$ comparison with $\sum_{n=1}^{\infty} \frac{1}{4^n}$ implies convergence.

* sum of two convergent series is convergent.