# Change of Variables in Mult. Int.

Electronic Course Instructor Survey
utdirect.utexas.edu/ctl/ecis (Friday)

Change of variables in = substitution

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

$$x = g(u) \qquad a = g(c)$$
$$dx = g'(u)\,du \qquad b = g(d)$$

Change variables in multiple integrals

$$\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\,\boxed{r}\,dr\,d\theta$$

we will explain this

Two systems of variables $\begin{cases} (x,y) \\ (r,\theta) \end{cases}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

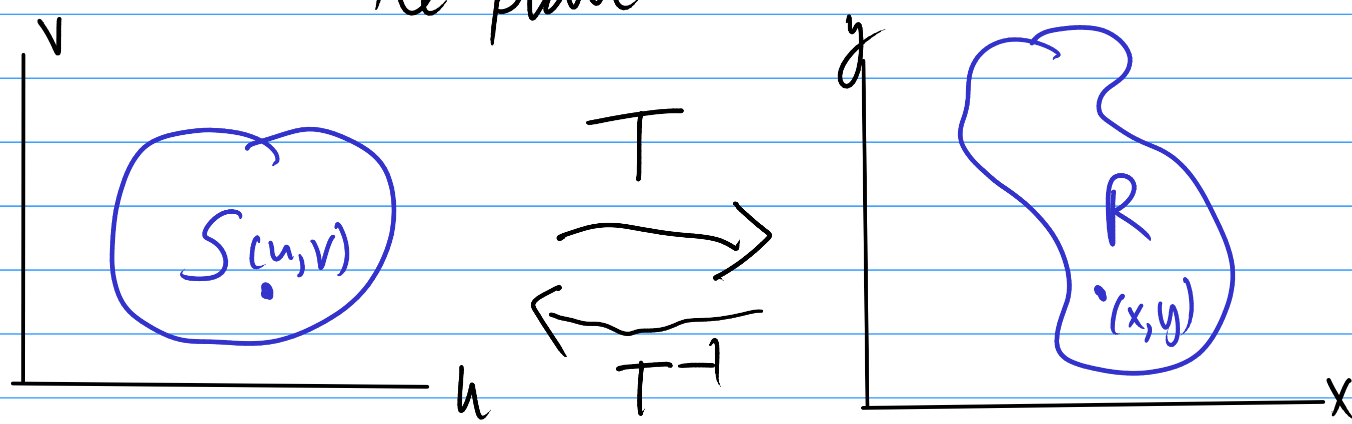
In general two systems of variables

$(x, y)$ $\quad$ $(u, v)$

$$T \begin{cases} x = g(u, v) \\ y = h(u, v) \end{cases}$$

$(x, y) = T(u, v)$ $\quad$ $T$ transformation

$T$ is a function whose domain and range are subsets of the plane




if $(x, y) = T(u, v)$

we say that $(x, y)$ is the image of $(u, v)$ under $T$.

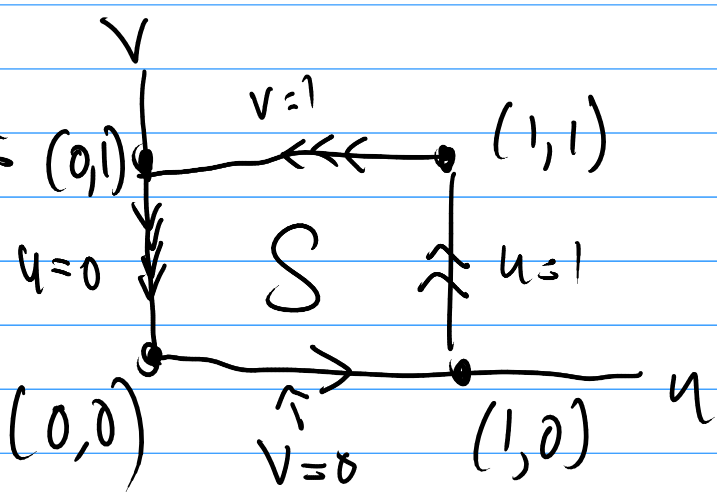
Image of a set $S$ is the set of all images of points in $S$

$T$ is one-to-one if no two points have the same image under $T$

In this case, it is possible to solve $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ for $u$ and $v$

$\left.\begin{cases} u = G(x,y) \\ v = H(x,y) \end{cases}\right\}$ Defines the inverse transformation $T^{-1}$

Ex




$\{x = u^2 - v^2 \quad y = 2uv\}$ $T$

What is the image of $S$?

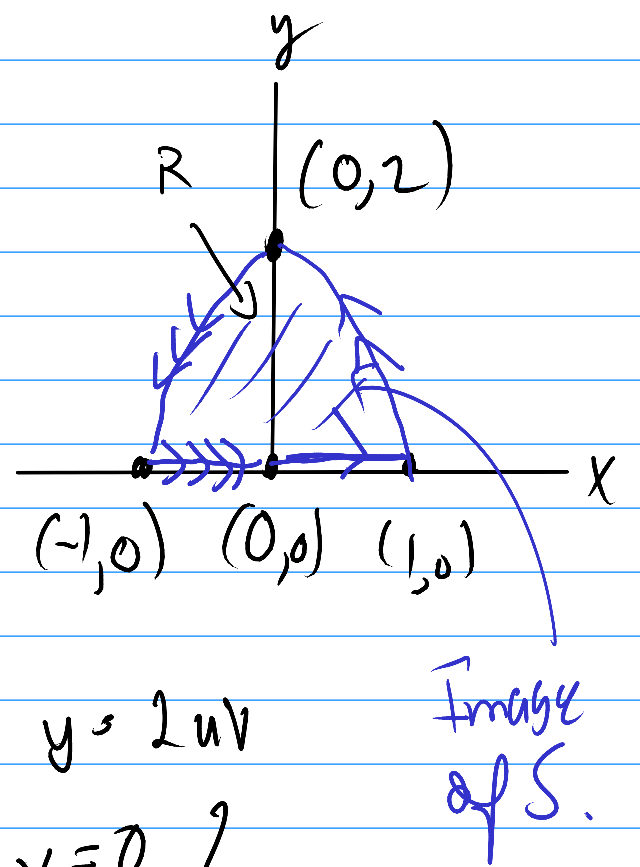
Start with corners

$T(0,0) = (0,0)$

$T(1,0) = (1,0)$

$T(0,1) = (-1,0)$

$T(1,1) = (0,2)$




Edges: $x = u^2 - v^2 \quad y = 2uv$

what happens to $v = 0$?

$v = 0 \Rightarrow x = u^2 \quad y = 0$

$x \geq 0$

$u = 0 \Rightarrow x = -v^2 \quad y = 0$

$x \leq 0$

$u = 1 \Rightarrow x = 1 - v^2 \quad y = 2v$

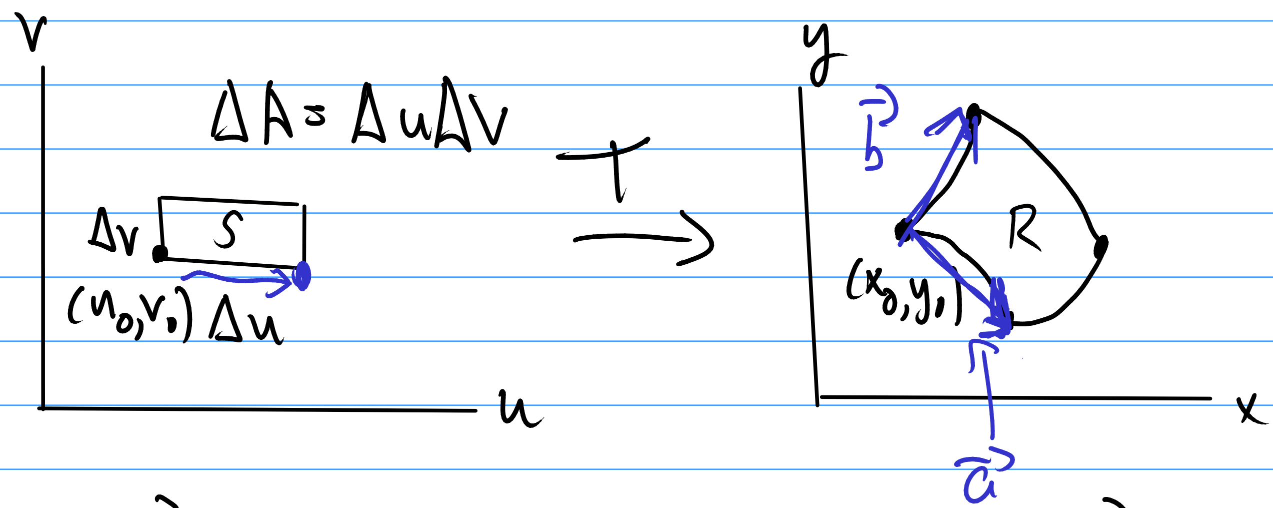
eliminate $v$: $x = 1 - \left(\frac{y}{2}\right)^2 = 1 - \frac{y^2}{4}$

$v = 1 \Rightarrow x = \frac{y^2}{4} - 1$

How does integral change

$$\iint f(u,v)\,\boxed{dA}$$

How does $dA$ change?




$$\vec{r}(u,v) = g(u,v)\vec{i} + h(u,v)\vec{j}$$

Approximation: $R \approx$ parallelogram

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0,v_0)$$

$$\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0,v_0)$$

$$\vec{r}(u_0+\Delta u, v_0) \approx \vec{r}(u_0,v_0) + \vec{r}_u \Delta u$$

$$\vec{r}(u_0, v_0+\Delta v) \approx \vec{r}(u_0,v_0) + \vec{r}_v \Delta v$$

$$\vec{a} = \vec{r}_u \Delta u$$

$$\vec{b} = \vec{r}_v \Delta v$$

Area of parallelogram

$$|\vec{a} \times \vec{b}| = |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}} \vec{k}$$

Jacobian
Determinant $\rightarrow \dfrac{\partial(x,y)}{\partial(u,v)}$
of $T$ at $(u_0, v_0)$

$$\Delta A = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$$

$$\Delta A = \Delta x \Delta y$$

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

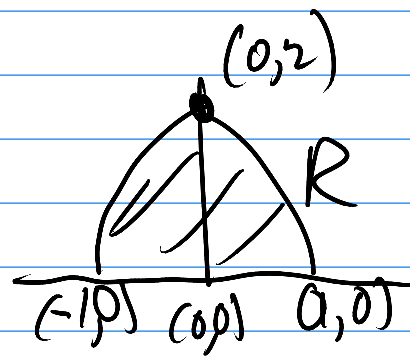
where $R$ is the image of $S$.

Ex polar coordinates

$$x = g(r,\theta) = r\cos\theta$$

$$y = h(r,\theta) = r\sin\theta$$

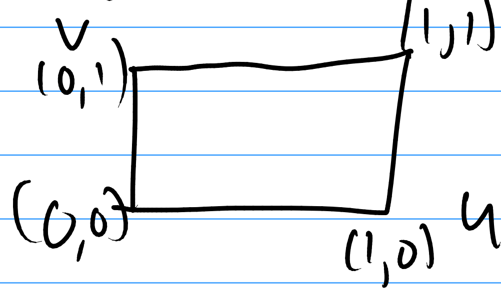
$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$




Ex $\displaystyle\iint_R y \, dA$

$$x = u^2 - v^2$$

$$y = 2uv$$

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$$\int_0^1\int_0^1 (2uv)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} = 2u & \frac{\partial y}{\partial u} = 2v \\ \frac{\partial x}{\partial v} = -2v & \frac{\partial y}{\partial v} = 2u \end{vmatrix} = (2u)^2 + (2v)^2$$

$$= 4(u^2+v^2)$$

$$\int_0^1\int_0^1 (2uv)\,4(u^2+v^2)\,du\,dv$$