The Chain Rule for Partial Derivatives

Actually different from single var

In this lecture, Leibniz notation is used throughout

$z = f(x,y)$ $\qquad \dfrac{\partial z}{\partial x} = f_x(x,y)$

$$\frac{\partial z}{\partial y} = f_y(x,y)$$

$y = f(x)$ $\qquad \dfrac{dy}{dx} = f'(x)$

1-var chain Rule $\qquad y = f(x)$

$$x = g(t)$$

y depends on t through x. $\quad y = f(g(t))$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(g(t))\, g'(t)$$

Simplest 2-variable case

$z = f(x,y)$, $x = g(t)$, $y = h(t)$

$z = f(g(t), h(t))$ composite function

"z depends on t through x and y."

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$ This is the chain rule

$$= f_x(g(t),h(t))\,g'(t) + f_y(g(t),h(t))\,h'(t)$$

<u>Example</u> $z = x^2y + 3xy^4$ $x = \sin 2t$

$y = \cos t$

$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

plug in functions of t for x and y

Pitfall: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Some people prove this formally by

$$\frac{dy}{\cancel{dx}}\frac{\cancel{dx}}{dt} = \frac{dy}{dt} \quad \text{QED??}$$

For multiple variables

$$\frac{\partial z}{\cancel{\partial x}}\frac{\cancel{dx}}{dt} \overset{?}{=} \frac{\partial z}{dt}\,? \neq \frac{dz}{dt}$$

Not true because you also need

$\frac{\partial z}{\partial y}\frac{dy}{dt}$ ← dependence of z on t <u>through</u> y.

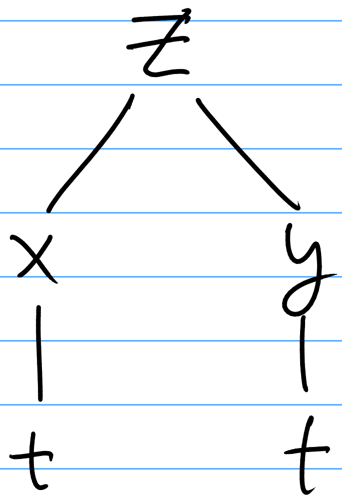
Similar to formula for differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

So don't cancel the differentials

Organize the dependencies between the variables in a "tree diagram" or dependency graph

$z = f(x,y) \qquad x = g(t) \qquad y = h(t)$



Each var depends on the vars below

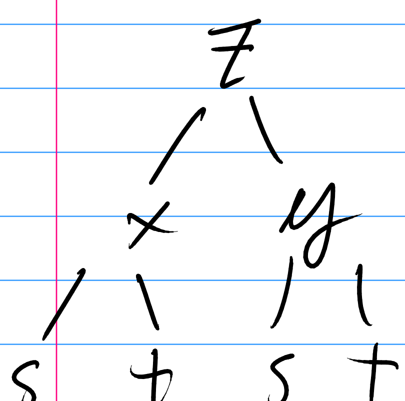
top = dependent variable

mid = intermediate

bottom = independent variable

A different case $z = f(x,y)$

$x = g(s,t) \qquad y = h(s,t)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex $z = e^x \sin y \quad x = st^2 \quad y = s^2 t$

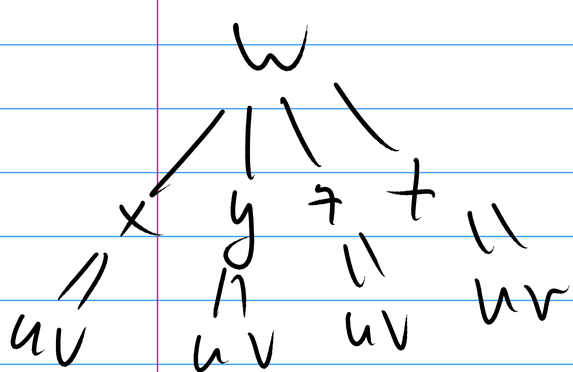
$$z = e^{st^2} \sin(s^2 t)$$

$$\frac{\partial z}{\partial s} = \left(e^x \sin y\right)\left(t^2\right) + \left(e^x \cos y\right)(2st)$$

$$= \left(e^{st^2} \sin(s^2 t)\right) t^2 + \left(e^{st^2} \cos s^2 t\right)(2st)$$

More complex $w = f(x, y, z, t)$

x, y, z, t are each functions of (u, v)



Chain rule computes $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$

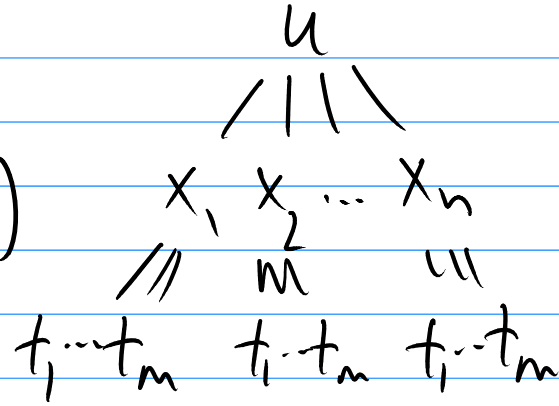
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

General Form

$$u = u(x_1, \dots, x_n)$$

$$x_i = x_i(t_1, \dots, t_m)$$



$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

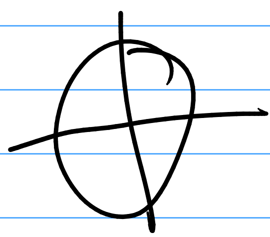
$$\vdots$$

$$\frac{\partial u}{\partial t_m} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Implicit differentiation

$F(x,y)=0$ defines y as a function of x, implicitly.

$$x^2+y^2=1 \Rightarrow y=\pm\sqrt{1-x^2}$$



Want to know $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(x^2+y^2=1\right) \Rightarrow 2x+2y\frac{dy}{dx}=0$$

solve for $\frac{dy}{dx} = -\frac{x}{y}$

Another derivation:

$$\frac{\partial}{\partial x}\left(F(x,y)=0\right)$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$$

Suppose $F(x,y,z)=0$ defines z as a function of x and y, implicitly

$\frac{\partial z}{\partial x}$: $\frac{\partial}{\partial x}\left(F(x,y,z)=0\right)$

(F is function of x, y, z and we regard x, y, z as functions of x, y)

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z}$$

(This works provided $F_z \neq 0$)

Ex $x^3 + y^3 + z^3 + 6xyz = 1$

Regard this as determining z as a function of x and y.

Want to know $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Implicit Differentiation:

$$\frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 6xyz\right) = \frac{\partial}{\partial x}(1) = 0$$

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6xy\frac{\partial z}{\partial x} + 6yz = 0$$

$$3x^2 + 6yz + (3z^2 + 6xy)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy}$$

Or, use chain rule

$$F(x, y, z) = x^3 + y^3 + z^3 + 6xyz$$

$$F_x = \frac{\partial F}{\partial x} = 3x^2 + 6yz$$

$$F_y = \frac{\partial F}{\partial y} = 3y^2 + 6xz$$

$$F_z = \frac{\partial F}{\partial z} = 3z^2 + 6xy$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy}$$