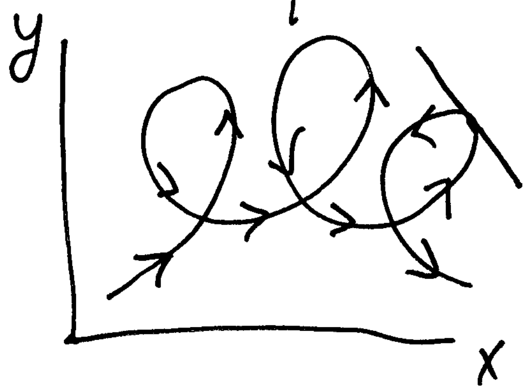
Calculus with parametric curves

Tangents, Areas, Arclengths.

Parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$



Tangents suppose you eliminate the parameter

$$y = F(x) \quad \text{want } \frac{dy}{dx} = F'(x)$$

$$y(t) = F(x(t))$$

$$y'(t) = F'(x(t))\, x'(t)$$

$$\frac{y'(t)}{x'(t)} = F'(x(t)) = \frac{dy}{dx}$$

$$y'(t) = \frac{dy}{dt}$$

$$x'(t) = \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

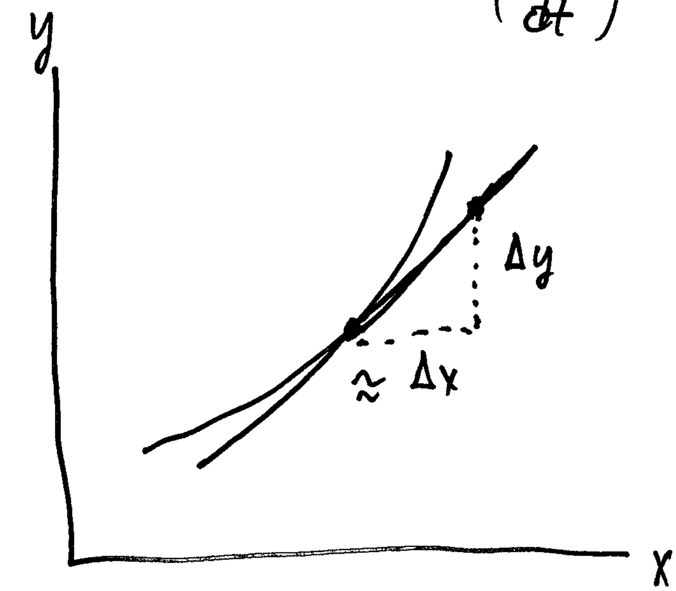
works provided that

$$x'(t) = \frac{dx}{dt} \neq 0$$

if $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ is the parametric equation for a curve

then the slope of the curve is given by

$$\text{slope} = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \text{if } \frac{dx}{dt} \neq 0.$$



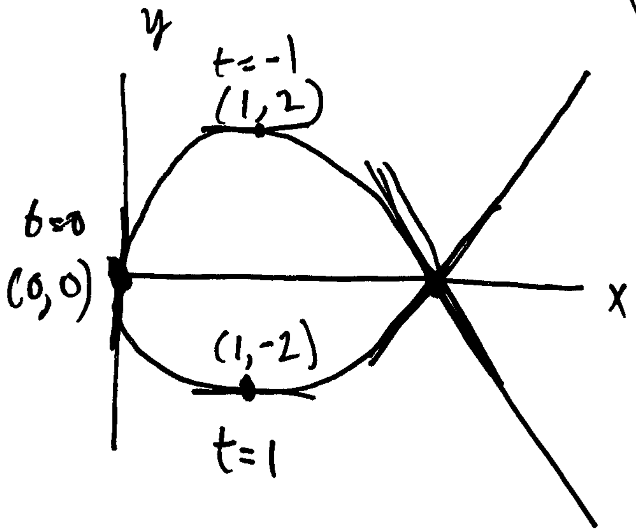
$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{\left(\frac{\Delta y}{\Delta t}\right)}{\left(\frac{\Delta x}{\Delta t}\right)} \xrightarrow{\Delta t \to 0} \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

If $\frac{dx}{dt} = 0$  vertical tangent.

If $\frac{dy}{dt} = 0$  horizontal tangent.

<u>Example</u> $C: \begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}$



Find the slope of the tangent line:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt} = 3t^2 - 3 \qquad \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

Horizontal tangents: $\frac{dy}{dt} = 0 = 3t^2 - 3$

$t^2 = 1 \quad t = \pm 1$

$t = +1 \quad x = 1, \; y = -2$ $\qquad t = -1 \quad x = 1, \; y = 2$

vertical tangents $\frac{dx}{dt} = 0 = 2t \qquad t = 0$

$t = 0 \quad x = 0, \; y = 0$

Self intersection at $(x,y) = (3,0)$

$$x = t^2 = 3 \qquad t = \pm\sqrt{3}$$

$$y(\pm\sqrt{3}) = 0$$

~~Thy~~ Slopes of tangent lines

at $t = -\sqrt{3}$ $\quad \dfrac{dy}{dx} = \dfrac{3}{2}\left(-\sqrt{3} - \dfrac{1}{-\sqrt{3}}\right)$

$$= \frac{3}{2}\left(\frac{-3}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \frac{-2}{\sqrt{3}}\,\frac{3}{2}$$

$$= -\sqrt{3}$$

at $t = \sqrt{3}$ $\quad \dfrac{dy}{dx} = \dfrac{3}{2}\left(\sqrt{3} - \dfrac{1}{\sqrt{3}}\right) = \sqrt{3}$

---

equation of the tangent line point-slope form.

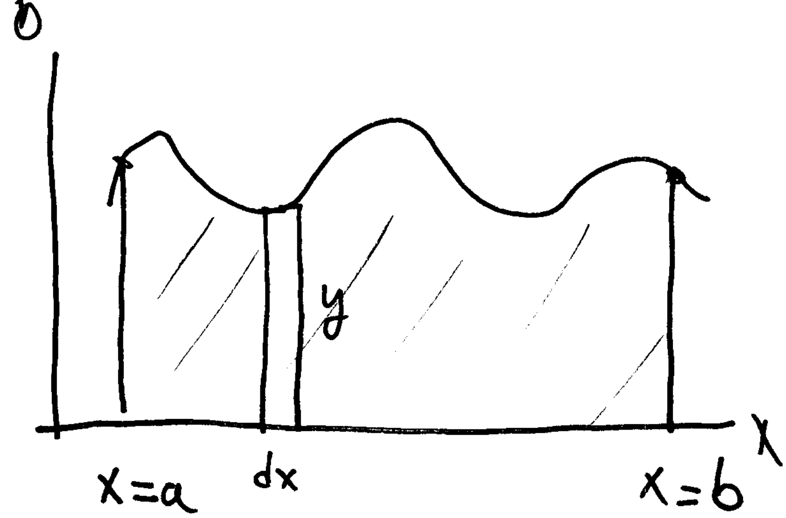
Equation of a line of slope m passing through (x_0, y_0) is

$$* \quad (y - y_0) = m(x - x_0)$$

$$\begin{cases} x_0 = x(t_0) \\ y_0 = y(t_0) \end{cases} \qquad \frac{dy}{dx} = \text{slope} \qquad y - y(t_0) = \frac{dy}{dx}\left(x - x(t_0)\right)$$

tangent line at $t = t_0$

Area under a parametric curve.



curve given by

$y = f(x)$

$$\text{Area} = \int_{x=a}^{x=b} f(x)\,dx$$

$$= \int_{x=a}^{x=b} y\,dx$$

just apply this to

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \qquad dx = f'(t)\,dt$$

$$\int_{x=a}^{x=b} y\,dx = \int_{t=\alpha}^{t=\beta} g(t)\,f'(t)\,dt$$

Tricky part: Limits of integration

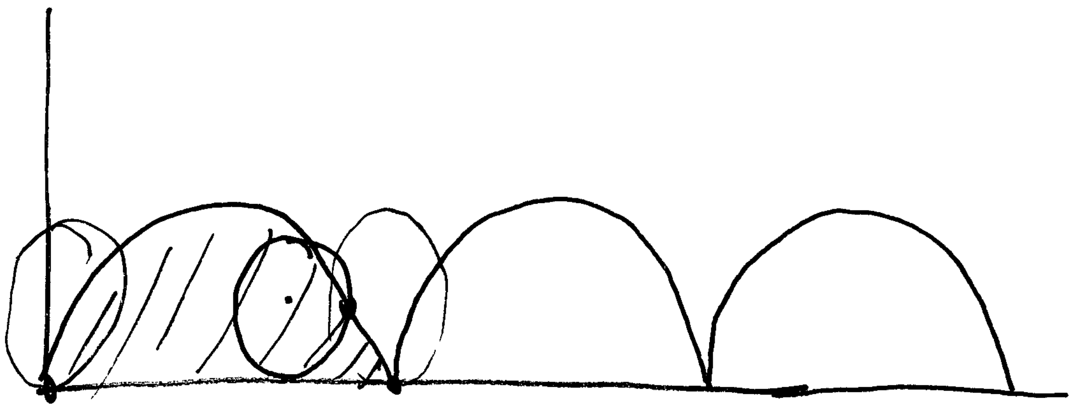
$t = \alpha$ is the "time" when $x = a$

$t = \beta$ is the "time" when $x = b$

$a = f(\alpha)$, $b = f(\beta)$

Find the area under one arch of the cycloid

PICTURE OF THE CYCLOID



$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{cases}$$

$dx = r(1-\cos\theta)\,d\theta$

$$\text{Area} = \int y\,dx = \int r(1-\cos\theta)\,r(1-\cos\theta)\,d\theta$$

one arch $\longleftrightarrow$ one rotation of the circle

$\longleftrightarrow$ $\theta = 0$ to $\theta = 2\pi$

$\longleftrightarrow$ $x = 0$ to $x = 2\pi r$

$$\text{Area} = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r^2(1-\cos\theta)^2\,d\theta$$

$$= r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta \right) d\theta$$

$$= r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2}\left(1 + \cos 2\theta\right) \right) d\theta$$

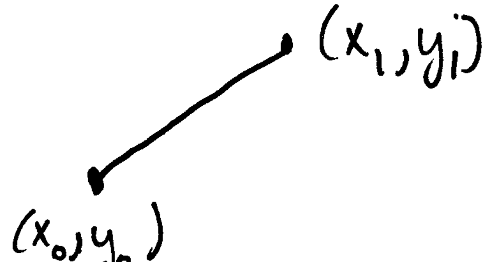
$$= r^2 \int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta \right) d\theta$$

$$= r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_{\theta=0}^{\theta=2\pi}$$

$$= r^2 \frac{3}{2}(2\pi) = 3\pi r^2$$

Arclength = length of a curve.

Length of a line segment.

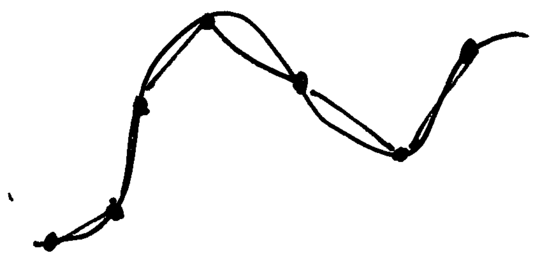


$$\Delta x = x_1 - x_0$$

$$\Delta y = y_1 - y_0$$

$$\text{Length} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

curve



approximate by line segments

Δx_i, Δy_i

run rise of each segment.

$$L = \sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

of course $\Delta x_i = \dfrac{\Delta x_i}{\Delta t_i} \Delta t_i$

$$L = \sum_i \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 (\Delta t_i)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 (\Delta t_i)^2}$$

$$= \sum_i \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

Take limits as $\Delta t_i \to 0$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

formula for arc length of a parametric curve.

<u>Ex</u> arc length of a circle of radius R

$x = R\cos t$

$y = R\sin t$

$$\underset{\alpha}{0} \le t \le \underset{\beta}{2\pi}$$

$$\frac{dx}{dt} = R(-\sin t) \qquad \frac{dy}{dt} = R\cos t$$

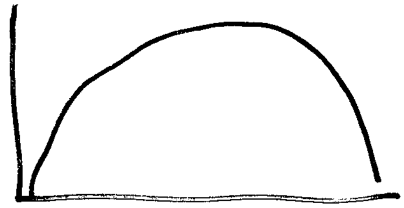
$$L = \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{R^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{R^2}\, dt = R\int_0^{2\pi} dt = 2\pi R$$

= circumference of circle

Arc length of cycloid



$$x = r(\theta - \sin\theta)$$

$$0 \le \theta \le 2\pi$$

$$y = r(1 - \cos\theta)$$

$$\frac{dx}{d\theta} = r(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = r\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{(r(1-\cos\theta))^2 + (r\sin\theta)^2}\, d\theta$$

$$= 8r$$