Derivatives and integrals of vector functions

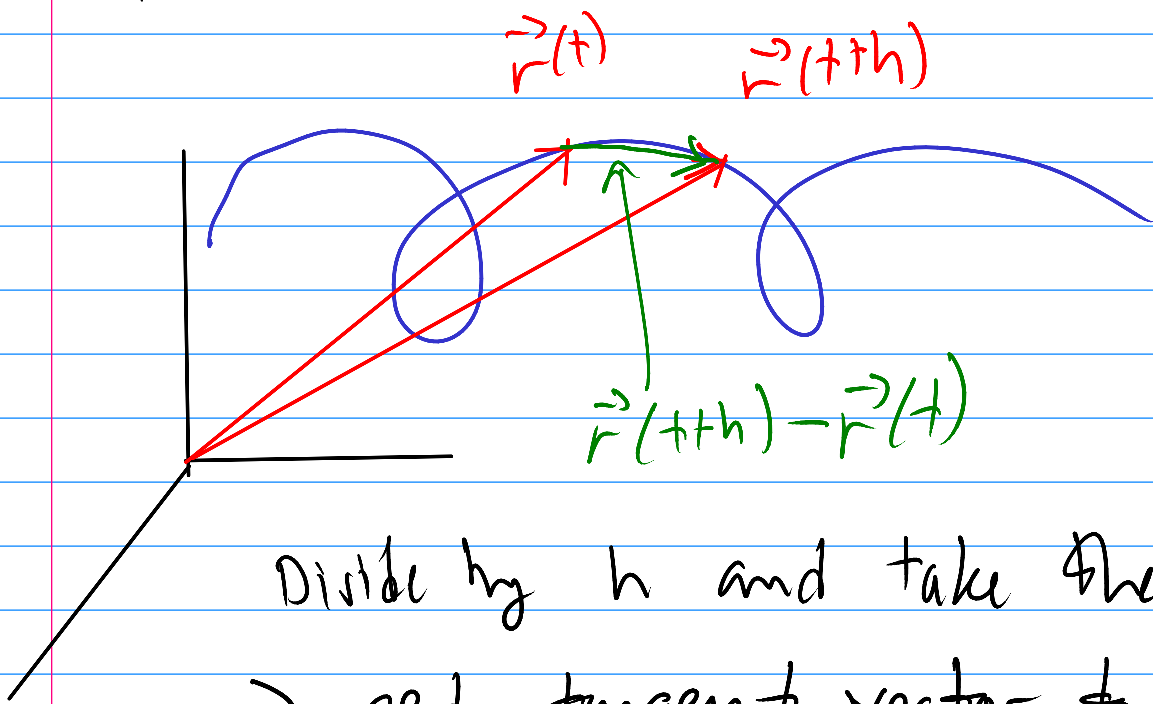
vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

one component at a time

$$\frac{d}{dt}\vec{r}(t) = \left(\frac{d}{dt}f(t)\right)\vec{i} + \left(\frac{d}{dt}g(t)\right)\vec{j} + \left(\frac{d}{dt}h(t)\right)\vec{k}$$

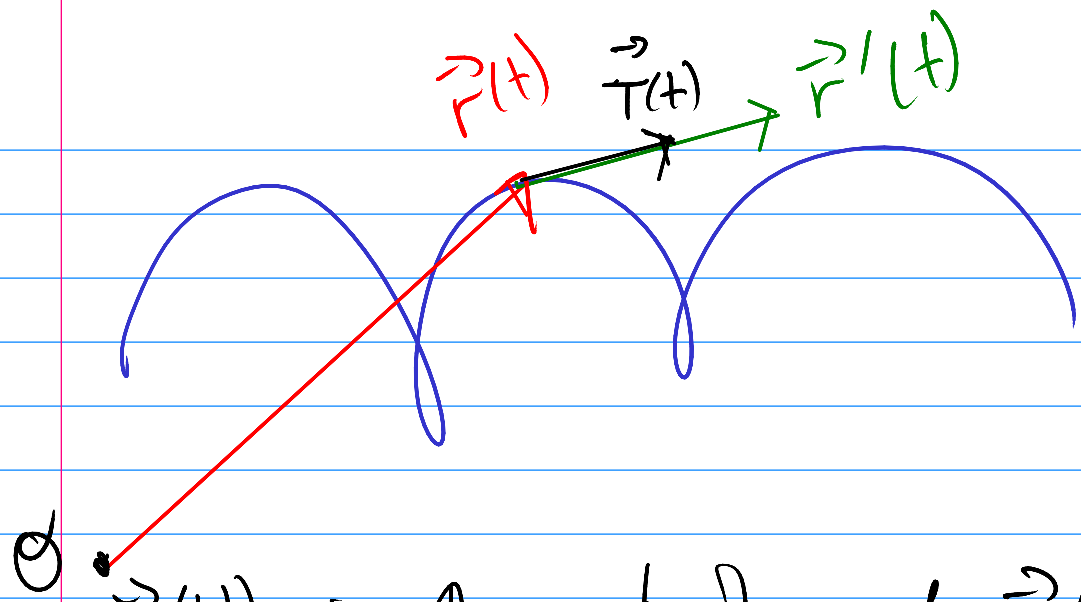
Recall $\displaystyle \frac{d}{dt}f(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$

$$\frac{d}{dt}\left[\vec{r}(t)\right] = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$



Divide by h and take the limit

$\Rightarrow$ get tangent vector to the curve



$\vec{r}\,'(t)$ is the velocity of $\vec{r}(t)$

$\|\vec{r}\,'(t)\|$ is the speed

direction of $\vec{r}\,'(t)$ = direction of motion

unit Tangent vector $= \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$\|\vec{T}(t)\| = 1$ direction of $\vec{T}(t)$ = direction of $\vec{r}\,'(t)$

assume $\vec{r}\,'(t) \neq 0$

$$\vec{T}(t) \cdot \|\vec{r}\,'(t)\| = \vec{r}\,'(t)$$

(unit tangent) (speed) = velocity

Find derivative and unit tangent

for $\vec{r}(t) = 4\sqrt{t}\,\vec{i} + t^2\vec{j} + t\vec{k}$

at $t = 1$ $\qquad \vec{r}(1) = \langle 4, 1, 1\rangle$

$$\frac{d}{dt}\vec{r}(t) = \frac{d}{dt}(4\sqrt{t})\vec{i} + \frac{d}{dt}(t^2)\vec{j} + \frac{d}{dt}(t)\vec{k}$$

$$= 2t^{-1/2}\vec{i} + 2t\vec{j} + 1\cdot\vec{k}$$

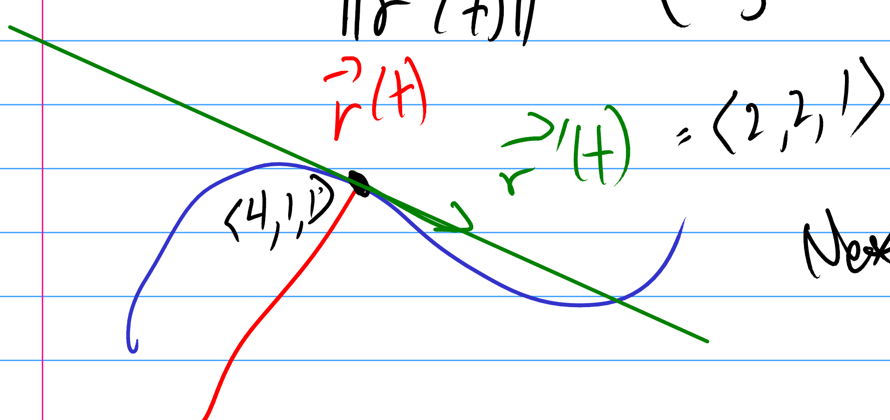
Plug in $t=1$

$$\vec{r}\,'(1) = 2\vec{i} + 2\vec{j} + 1k = \langle 2, 2, 1\rangle$$

unit tangent vector

$$\|\vec{r}\,'(1)\| = \sqrt{(2)^2 + (2)^2 + 1^2} = \sqrt{9} = 3$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$



Next, Find this line

Point $= \vec{r}(1)$, direction $= \vec{r}\,'(1)$

$\vec{A}(s) = \vec{r}(1) + s\,\vec{r}\,'(1)$ $\quad$ $\vec{A}(s)$ parameterizes the line.

$= \langle 4,1,1\rangle + s\langle 2,2,1\rangle$

$= \langle 4+2s,\ 1+2s,\ 1+s\rangle$

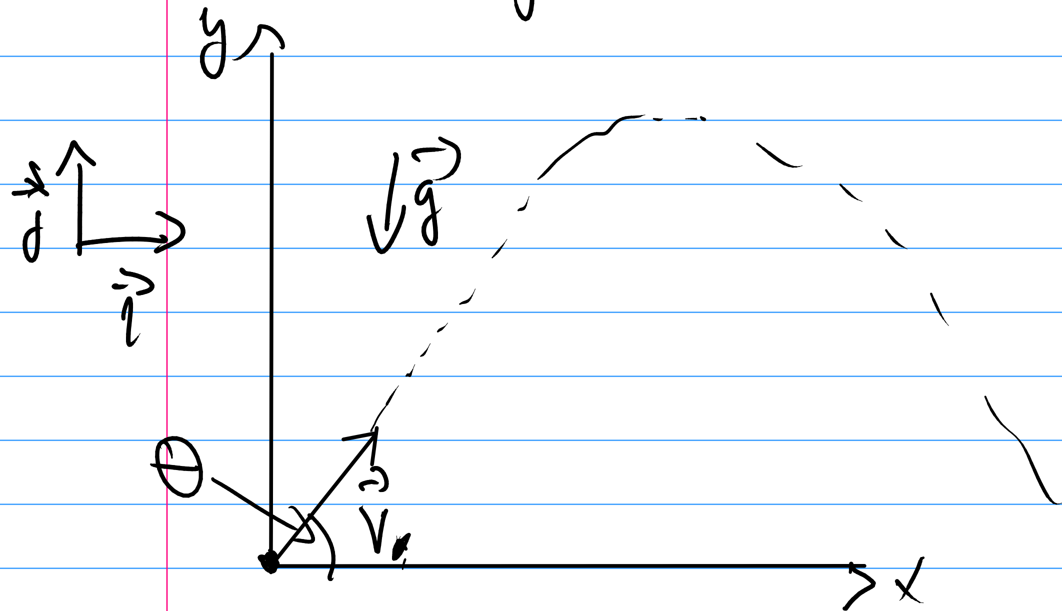
Revisit constant force example

$\vec{r}(t)$ position

$\vec{v}(t) = \vec{r}\,'(t)$ velocity $\quad$ constant force

$\vec{a}(t) = \vec{r}\,''(t)$ acceleration $\quad$ $\vec{F} = m\vec{a}$

Constant gravitational field $\vec{g}$



$\vec{F} = m\vec{g}$

$= mg(-\vec{j})$

Know acceleration $\vec{a} = \frac{\vec{F}}{m} = -g\vec{j}$

Get velocity by integrating the acceleration

$$\int_0^t \vec{a}(s)\,ds = \int_0^t -g\vec{j}\,ds = -g\vec{j}\int_0^t s$$

$$= -gt\vec{j}$$

$$\vec{v}(t) - \vec{v}(0)$$

$$= \vec{v}(t) - \vec{v}_0$$

$$\vec{v}(t) = \vec{v}_0 - gt\vec{j}$$

get position by integrating velocity

$$\int_0^t \vec{v}(s)\,ds = \int_0^t (\vec{v}_0 - gs\vec{j})\,ds$$

$$= \vec{v}_0 t - \frac{1}{2}t^2 g\vec{j}$$

$$\vec{r}(t) - \cancel{\vec{r}(0)}$$

$$\vec{r}(t) = \vec{v}_0 t - \frac{1}{2}t^2 g\vec{j}$$

Write in $\vec{i}, \vec{j}$ components

$$\vec{v}_0 = \|\vec{v}_0\| \left(\cos\theta \, \vec{i} + \sin\theta \, \vec{j} \right)$$

$$\vec{r}(t) = \left(\|\vec{v}_0\| \cos\theta \cdot t \right) \vec{i} + \left(\|\vec{v}_0\| \sin\theta \cdot t - \frac{1}{2} t^2 g \right) \vec{j}$$

$$x = \|\vec{v}_0\| \cos\theta \; t$$

$$y = \|v_0\| \sin\theta \; t - \frac{1}{2} t^2 g$$

Eliminate The parameter:

$$t = \frac{x}{\|v_0\| \cos\theta}$$

$$y = \frac{\|v_0\| \sin\theta}{\|v_0\| \cos\theta} x - \frac{1}{2} \left(\frac{x}{\|v_0\| \cos\theta} \right)^2 g$$

$$= \tan\theta \, x - \frac{1}{2} \left(\frac{x}{\|v_0\| \cos\theta} \right)^2 g$$