Alternating Series

Serveral of the test we've developed so far, such as the integral test, comparison tests only work for series with positive terms.

What to do if series $\sum_{n=1}^{\infty} a_n$ has some $a_n > 0$ and some $a_n < 0$?

Most basic test: Absolute convergence

Given $\sum_{n=1}^{\infty} a_n$, consider $\sum_{n=1}^{\infty} |a_n|$, which is a sequence with positive terms.

Theorem (Absolute convergence $\Longrightarrow$ convergence)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Terminology: If $\sum_{n=1}^{\infty} |a_n|$ converges, we say $\sum_{n=1}^{\infty} a_n$ converges absolutely / is absolutely convergent

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

series has terms which alternate in sign

$\rightarrow$ can't use $\int$ test, comparison difficult.

BUT look at $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

which converges by p-test!

So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent, hence convergent.

Proof that absolute convergence implies convergence.

Fact $\quad 0 \leq a_n + |a_n| \leq 2|a_n|$

If $\sum |a_n|$ converges, $\sum 2|a_n|$ converges

by comparison test $\quad \sum (a_n + |a_n|)$ converges

Now $\quad \sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$

is the difference of two convergent series,

and hence $\sum a_n$ is convergent.

what about $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

Try absolute convergence : $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

divergent by p-test !

So the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is NOT absolutely convergent.

* But the fact is, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

* To see this, notice that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

is an alternating series : the terms alternate between positive and negative.

* If $\sum a_n$ is an alternating series, we can write

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n ,$$

where $b_n = |a_n|$ is a positive sequence.

An alternating series always has a lot of cancellation between the terms, that makes it "more likely" for the series to converge. This is expressed precisely by the following

<u>Theorem</u> If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 \cdots \quad (b_n > 0)$$

satisfies (i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

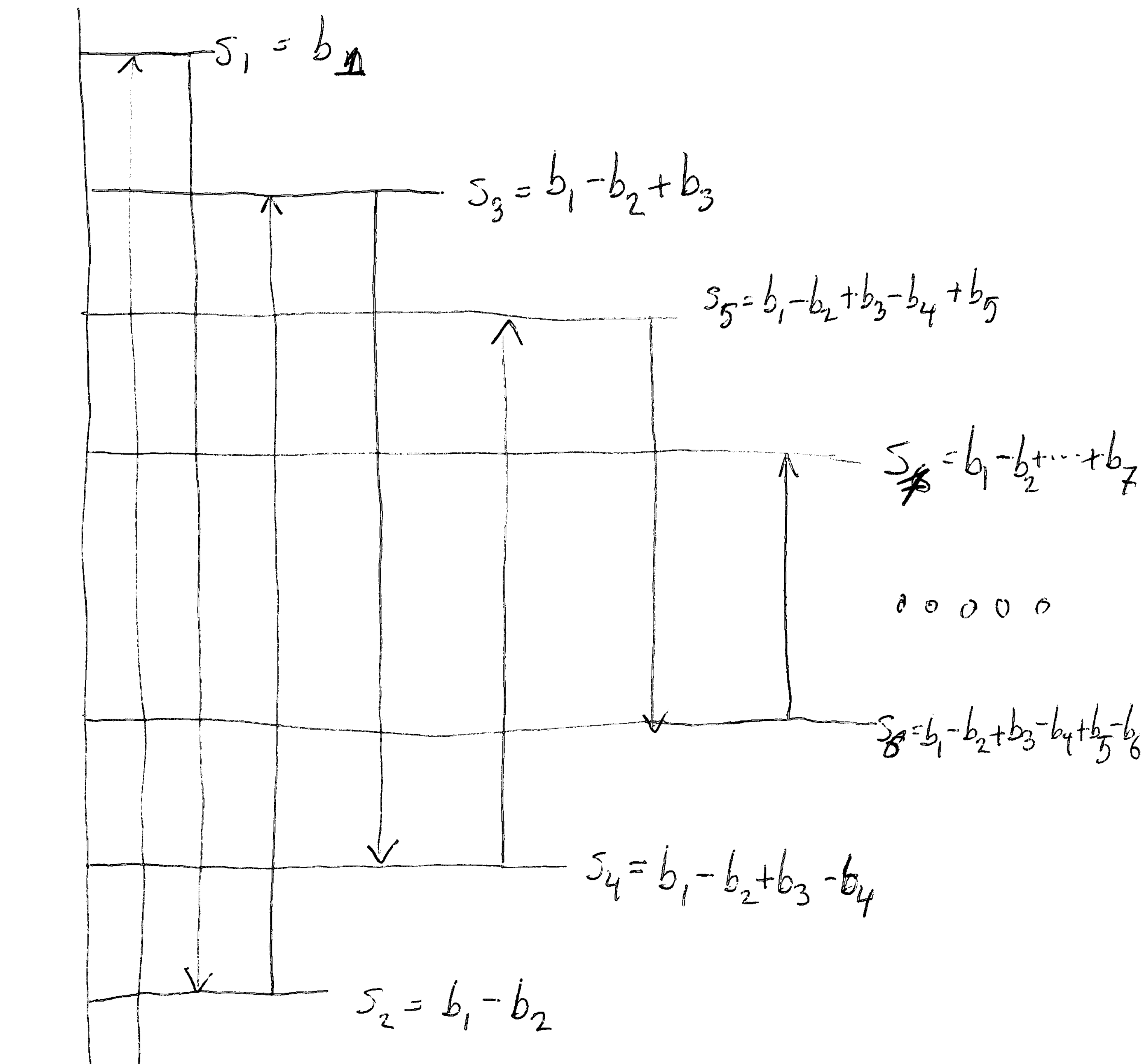
THEN the series is <u>convergent</u>.

This implies that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, since $b_n = \frac{1}{n}$

(i) $b_{n+1} \leq b_n$ means $\frac{1}{n+1} \leq \frac{1}{n}$ ✓check

(ii) $\lim_{n\to\infty} b_n = 0$ means $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓check.

why is alternating series test true: lots of cancellation



(i) $b_{n+1} \leq b_n$ implies the picture is accurate

(ii) $\lim_{n\to\infty} b_n = 0$ implies the process stabilizes.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)} = \frac{1}{\ln(5)} - \frac{1}{\ln(6)} + \cdots$

alternating with $b_n = \frac{1}{\ln(n+4)}$

decreasing? $b_{n+1} \leq b_n$ yes because $\ln(n+4)$ is an increasing function, and it's reciprocal is therefore decreasing.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln(n+4)} = \frac{1}{\infty} = 0$ ✓

so both conditions are satisfied, and

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$ converges.

Note If we look for absolute convergence, we find that $\sum_{n=1}^{\infty} \frac{1}{\ln(n+4)}$ diverges

by comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$:

$\frac{1}{\ln(n+4)} > \frac{1}{n}$ for $n \geq 2$.

<u>Ex</u> $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ Alternating with $b_n = \frac{3n-1}{2n+1}$

But $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0.$

Alternating series test does not apply

In this situation, the "test for divergence" immediately implies that

$\sum_{n=1}^{\infty} (-1)^n b_n$ <u>diverges</u> if $\lim_{n\to\infty} b_n \neq 0.$

So $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ <u>diverges</u>.

Ex $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ alternating with $b_n = \dfrac{n}{\sqrt{n^3+2}}$

check limit: $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+\frac{2}{n^2}}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\sqrt{1+\frac{2}{n^3}}} = \frac{1}{\infty \cdot \sqrt{1+0}} = \frac{1}{\infty} = 0$$

decreasing? A little hard to see directly: use derivative:

$$f(x) = \frac{x}{\sqrt{x^3+2}}, \quad f'(x) = \frac{\sqrt{x^3+2} - x\,(x^3+2)^{-1/2}\left(\frac{1}{2}\right)\cdot 3x^2}{x^3+2}$$

want to see where $f'(x) \leq 0$ i.e., where is f decreasing?

Denominator is positive, so forget it.

$$\sqrt{x^3+2} - \frac{3}{2}x^3(x^3+2)^{-1/2} \leq 0$$

$$\sqrt{x^3+2} \leq \frac{3}{2}x^3(x^3+2)^{-1/2}$$

$$x^3+2 \leq \frac{3}{2}x^3$$

$$2 \leq \frac{1}{2}x^3$$

$$4 \leq x^3$$

$$\sqrt[3]{4} \leq x$$

So $f(x)$ is decreasing as soon as $x \geq \sqrt[3]{4}$.

Good enough for alternating series test.

$\Rightarrow$ Get convergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$$

It is quite easy to estimate the error of the partial sums of an alternating series "The error is less than the first neglected term".

Theorem Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is alternating series such that $0 \le b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$.

These hypotheses imply $\sum(-1)^{n-1}b_n$ is convergent, so let $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ be the sum.

Let $S_n = \sum_{i=1}^{n}(-1)^{i-1}b_i = b_1 - b_2 + b_3 - \cdots + (-1)^{n-1}b_n$

be the nth partial sum.

The error in the approximation $S \approx S_n$ is $|S - S_n|$, and we have

$$\boxed{|S - S_n| \le b_{n+1}}$$

(b_{n+1} is the first term that S_n neglects)

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ How many terms in the partial sum do we need to take in order to garantee that

$|error| = |s - s_n| \leq 0.000001$

$= (\text{one millionth})$

$b_n = \frac{1}{n^6}$ Know $|s - s_n| \leq b_{n+1} = \frac{1}{(n+1)^6}$

~~So~~ Find smallest $(n+1)$ such that $\frac{1}{(n+1)^6} \leq 0.000001$

Ans: since $0.000001 = 10^{-6}$,

we can take $n+1 = 10$, and we get $\frac{1}{(n+1)^6} = \frac{1}{10^6} = 0.000001$

So we need the first $n=9$ terms. in order to get the desired accuracy:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} = 1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \frac{1}{8^6} + \frac{1}{9^6}$$

$$\pm 0.000001$$