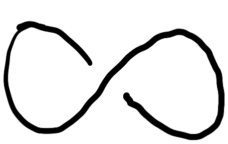
YSP Knots July 2, 2019

Time for some definitions.

<u>Def</u>: A <u>Knot</u> is a circular loop embedded in 3-dimensional space.

The key word is "embedded." This means the loop does not intersect itself in 3D space

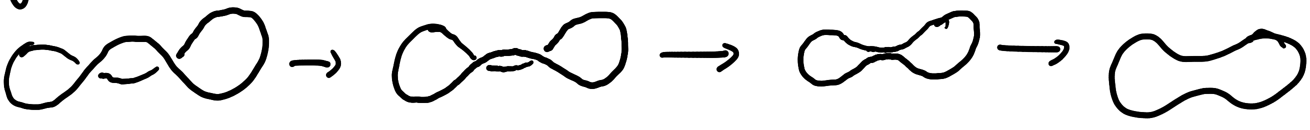
Good: 

Bad: 

<u>Def</u> Two knots are <u>equivalent</u> (or "the same") if it is possible to move one to the other in 3D space without cutting, gluing, or creating self-intersections. We <u>do</u> allow the string to stretch or shrink, however.

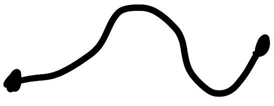
Two knots are equivalent if there is a "movie" that starts at one, ends at the other, and which never stops being a knot.

e.g.



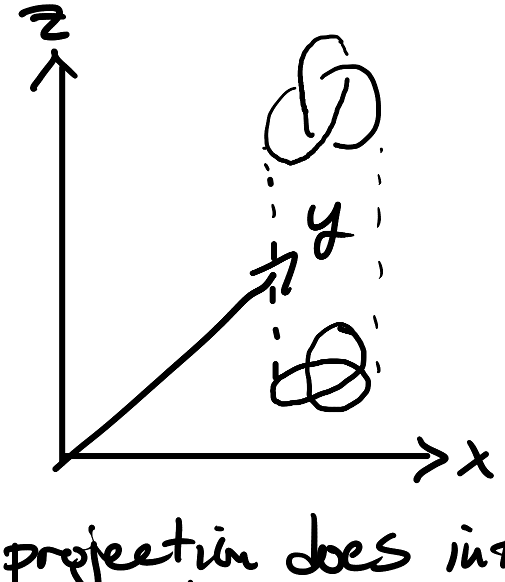
Definition: A link is defined similarly to a knot, but it involves multiple loops, called the components of the link. Equivalence of links is defined similarly to that of knots

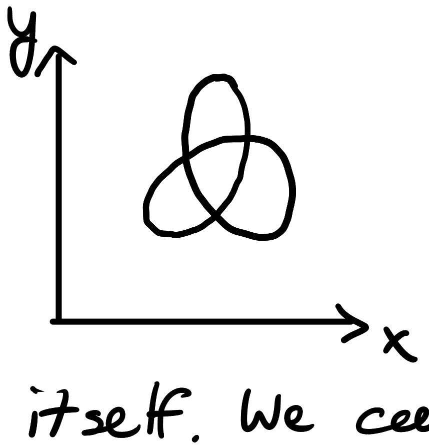
Note: "Knot" = "link with 1 component"

Note: If we try to use open strings  we don't get anything interesting, unless we fix the ends somehow (more on this later)

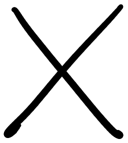
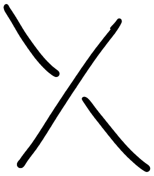
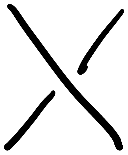
Projections and diagrams

A knot/link is a 3D object, but we can draw a 2D picture of it by projecting onto a plane:

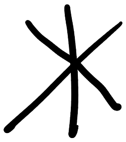
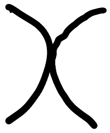
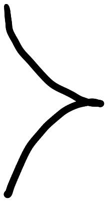




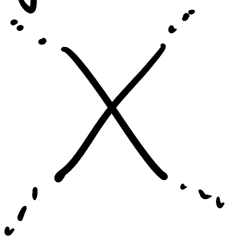
The projection does intersect itself. We call these points crossings. At each crossing, we draw it so as to indicate which strand was above and which below.

So  becomes  or 

Semi-technical remark: It is possible for other things to happen in the knot projection, such as

- 3 strands meet at same point 
- 2 strands tangent 
- Strand has tangent line parallel to projection axis 

We say a projection is *good* if these don't happen, and everywhere the projection looks like

 or 

Lemma: every knot has a good projection.

Exercise: convince yourself of this.
Hint: Just "wiggle" the knot to get rid of "bad" features.

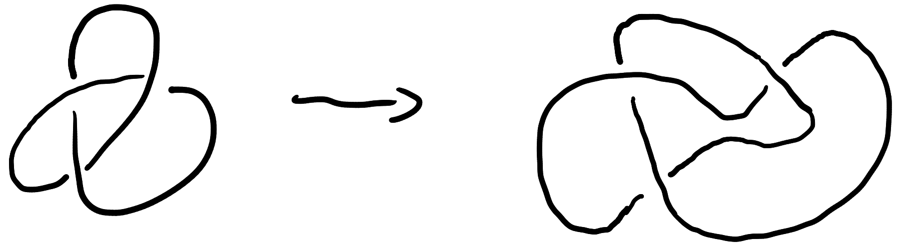
Def: A good knot projection with understrand/overstrand markings is called a *knot diagram*.

Basic issue: There are (infinitely) many knot diagrams that represent the same knot.

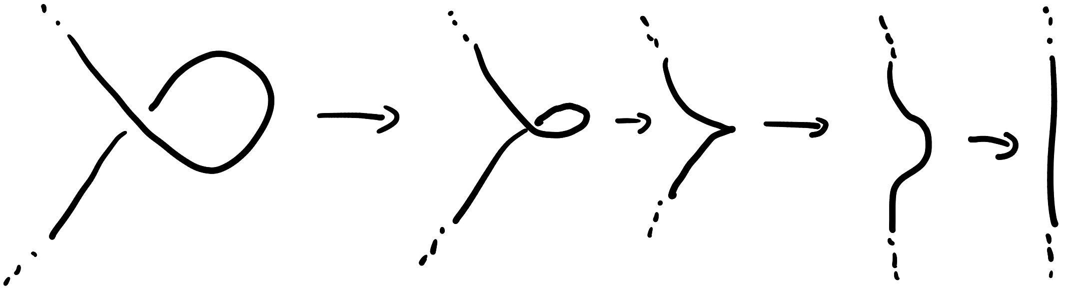
What transformations can we perform on a knot diagram that don't change the knot?

"Planar isotopy": Moving bits of the diagram without changing the pattern of strands and crossings.

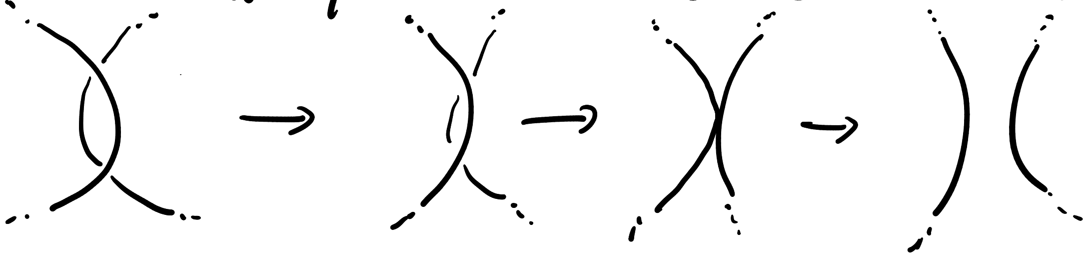
Eg.



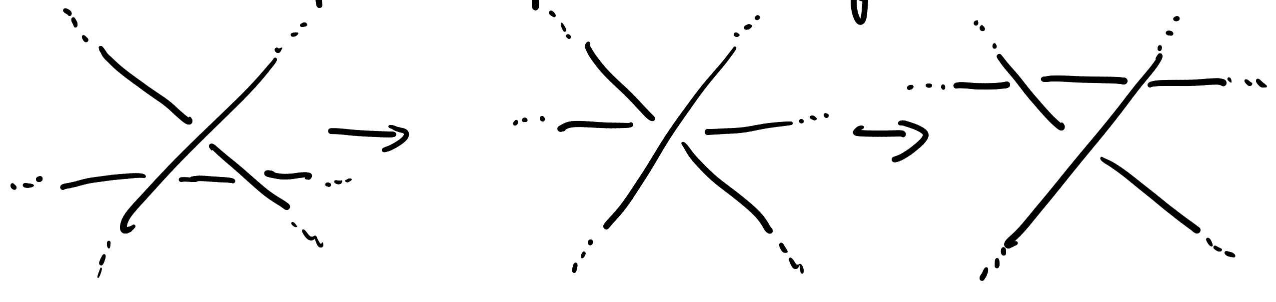
Teardrop removal/formation (Reidemeister I)



Moon removal/formation (Reidemeister II)



Strand slides past independent crossing (Reidemeister III)



YSP Knots

July 2, 2019

1. Classify all knots that can have a diagram with one crossing.

2. Classify all knots that can have a diagram with two crossings.

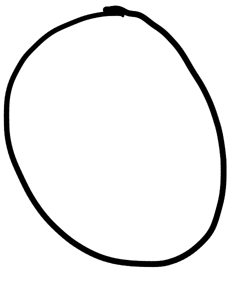
3. Classify all knots that can have a diagram with three crossings.

4. Same questions as 1-3 but with 2-component links instead of knots.

5. Consider the following list of knot/link diagrams. Try and see if you can figure out which diagrams represent equivalent knots/links.

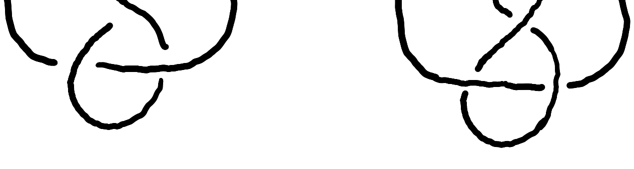
   For each pair of equivalent diagrams, write out a sequence of Reidemeister moves that takes one diagram to the other.

Unknot

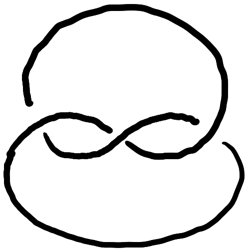


Trefoils

Figure 8 knots

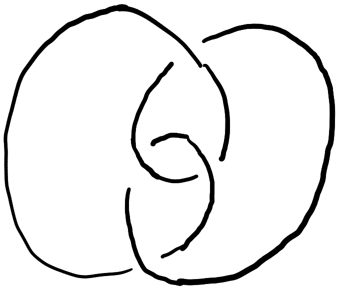


Overhand knots



Others:

Ⓐ



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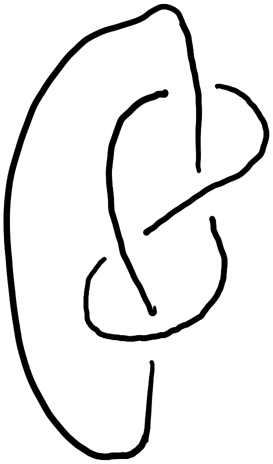

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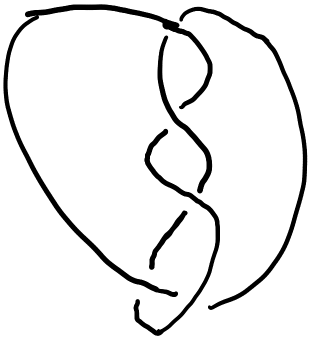

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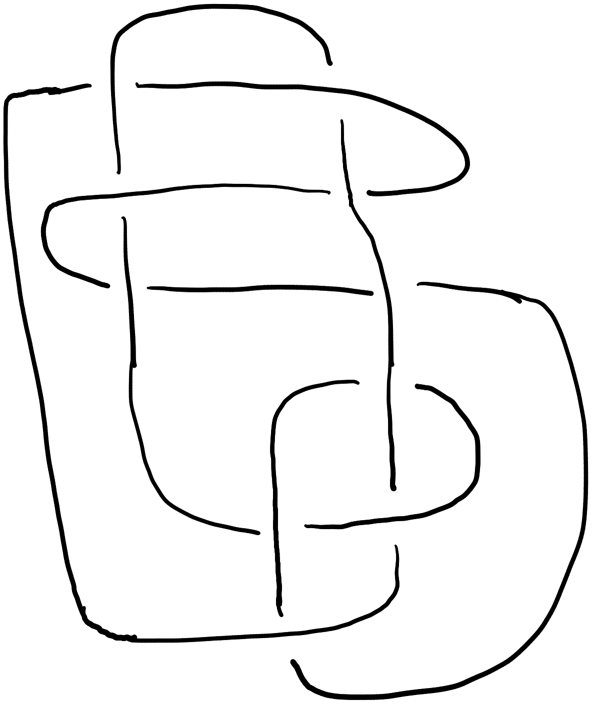

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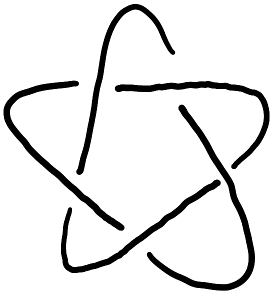

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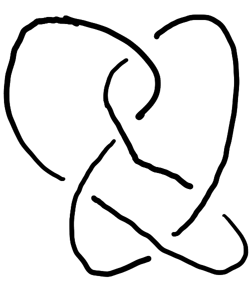
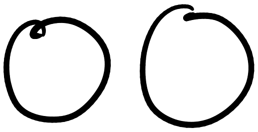


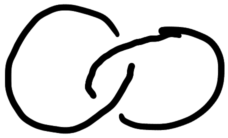
General Question:
What happens if you switch 1 crossing?

2-component links

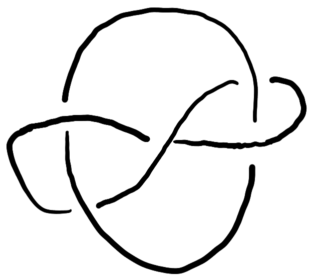
Unlink



Hopf link



Whitehead link



Others

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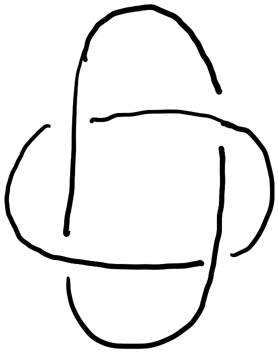
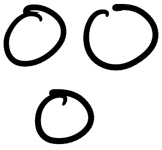


3-component links

Unlink



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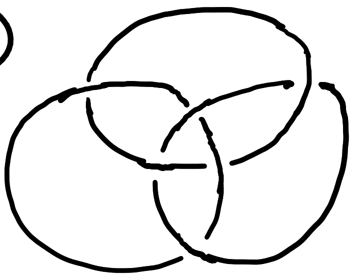

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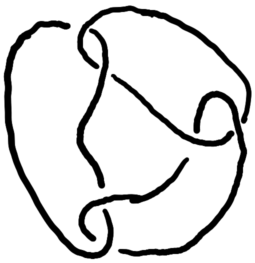

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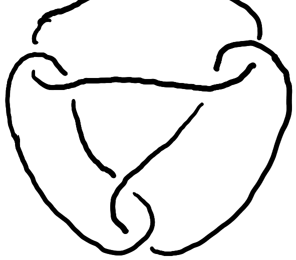


YSP Knots

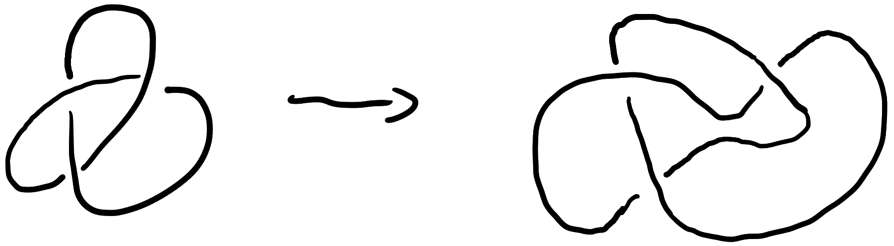
July 3, 2019

<u>Basic issue</u>: There are (infinitely) many knot diagrams that represent the same knot.

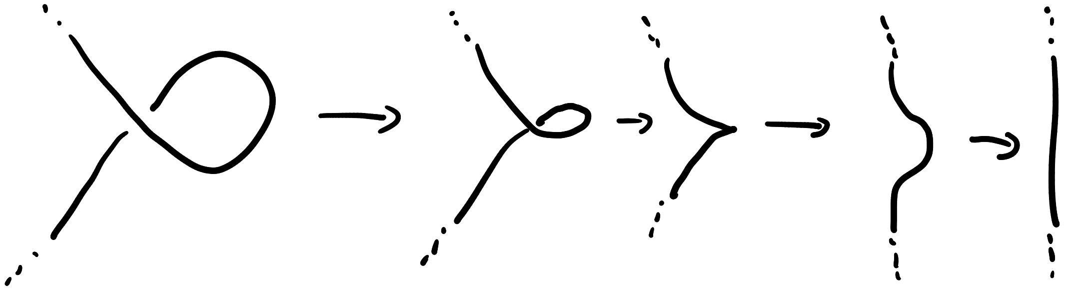
What transformations can we perform on a knot diagram that don't change the knot?

"Planar isotopy": Moving bits of the diagram with changing the pattern of strands and crossings.

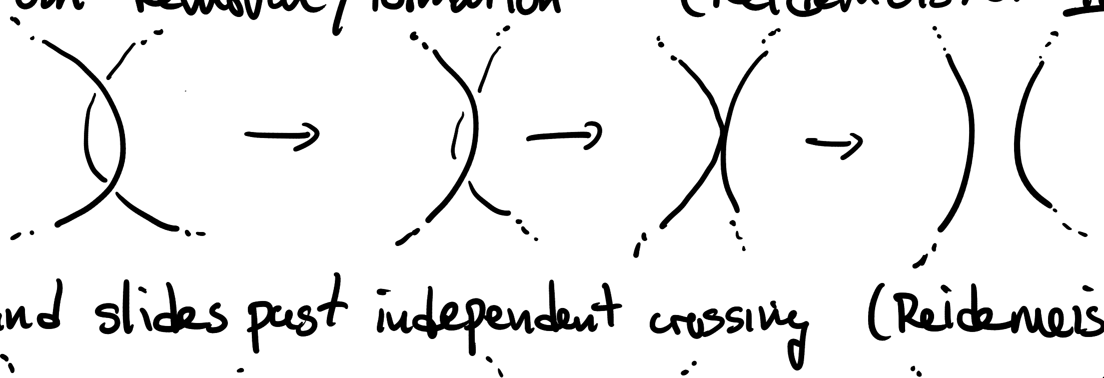
Eg.



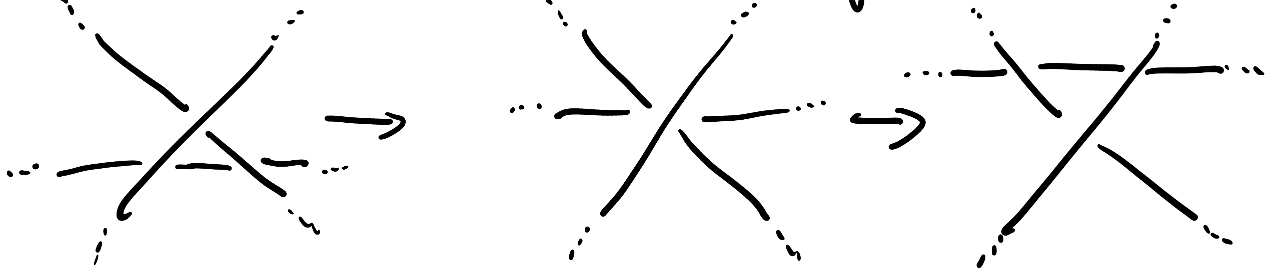
Teardrop removal/formation (Reidemeister I)



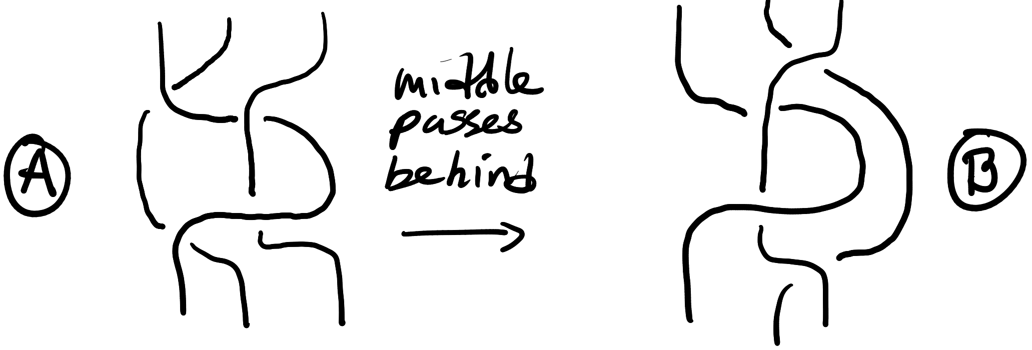
Moon removal/formation (Reidemeister II)



Strand slides past independent crossing (Reidemeister III)

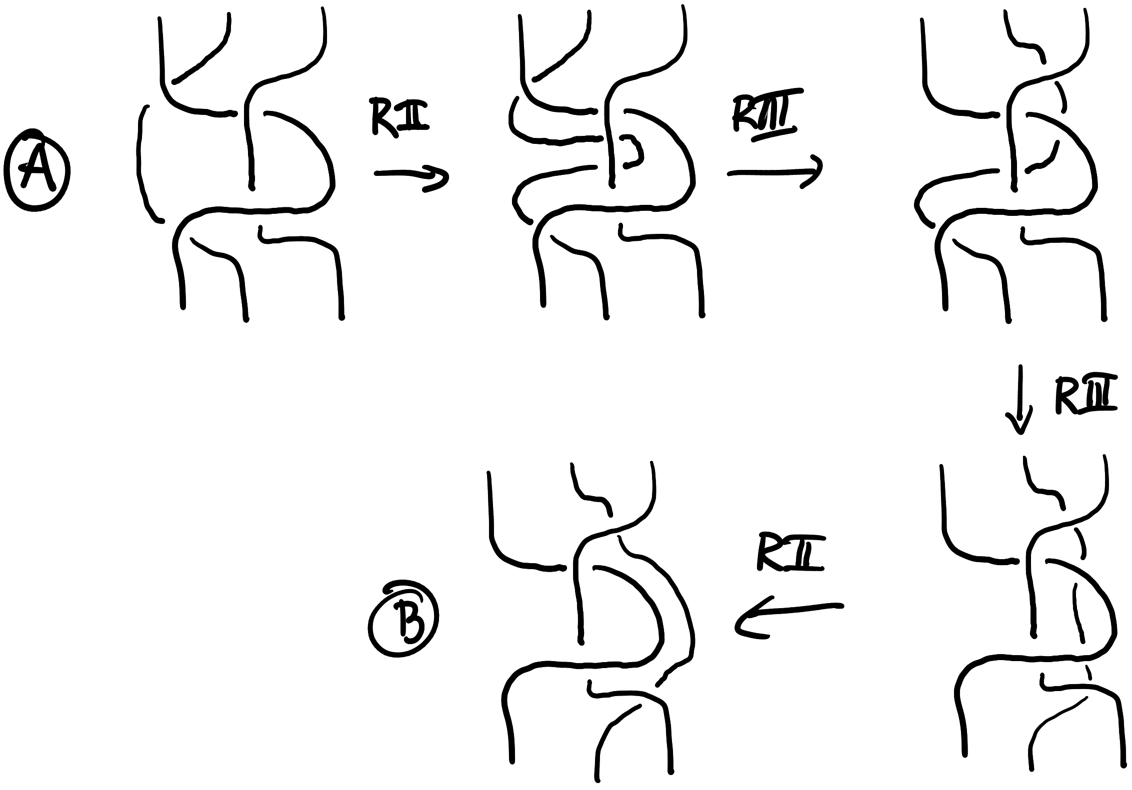


These three moves are known as Reidemeister moves. There are many more moves that are possible. For instance consider



Do we need to add this as a new move?

In fact, this move can be seen as a sequence of Reidemeister moves



Exercise: Come up with another "move". Find a sequence of Reidemeister moves that has the same effect.

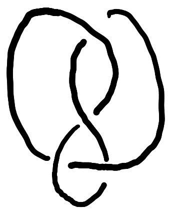
Reidemeister Theorem: Reidemeister moves suffice to connect any pair of equivalent knot diagrams. That is, for any "move" we can do on a knot diagram (that doesn't change the knot), there is a sequence of Reidemeister moves that has the same effect.

Problem: Prove Reidemeister's theorem.*

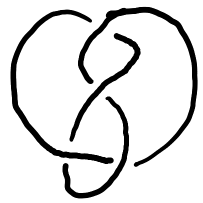
*Give an argument you find convincing.

Hint: Only move a small piece of the knot at a time.

Recall Figure 8 knot:

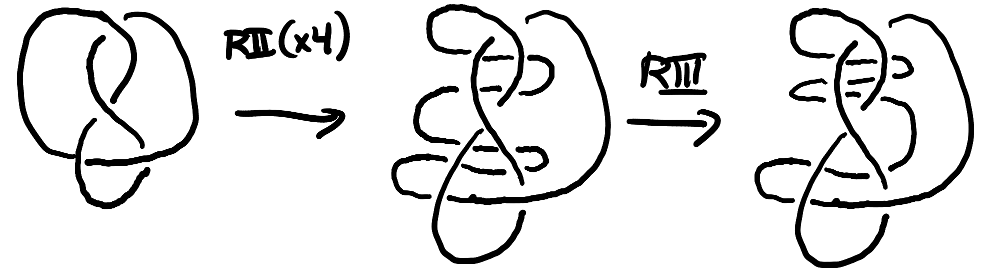


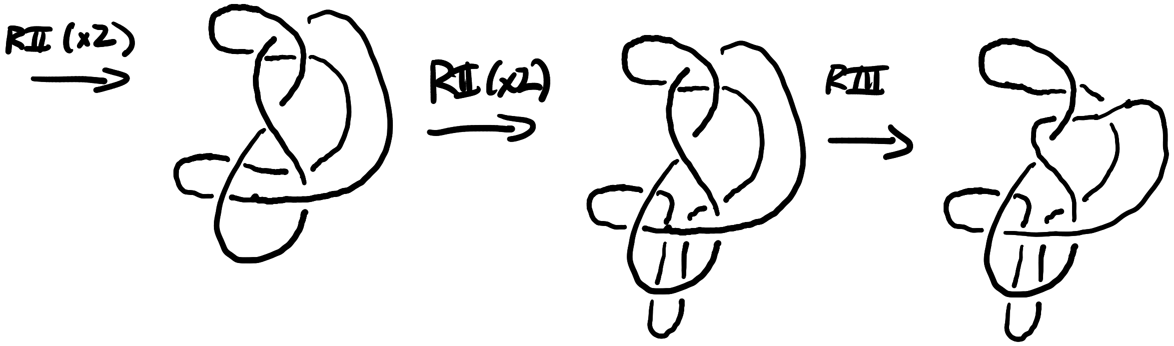
and "mirror Figure 8"

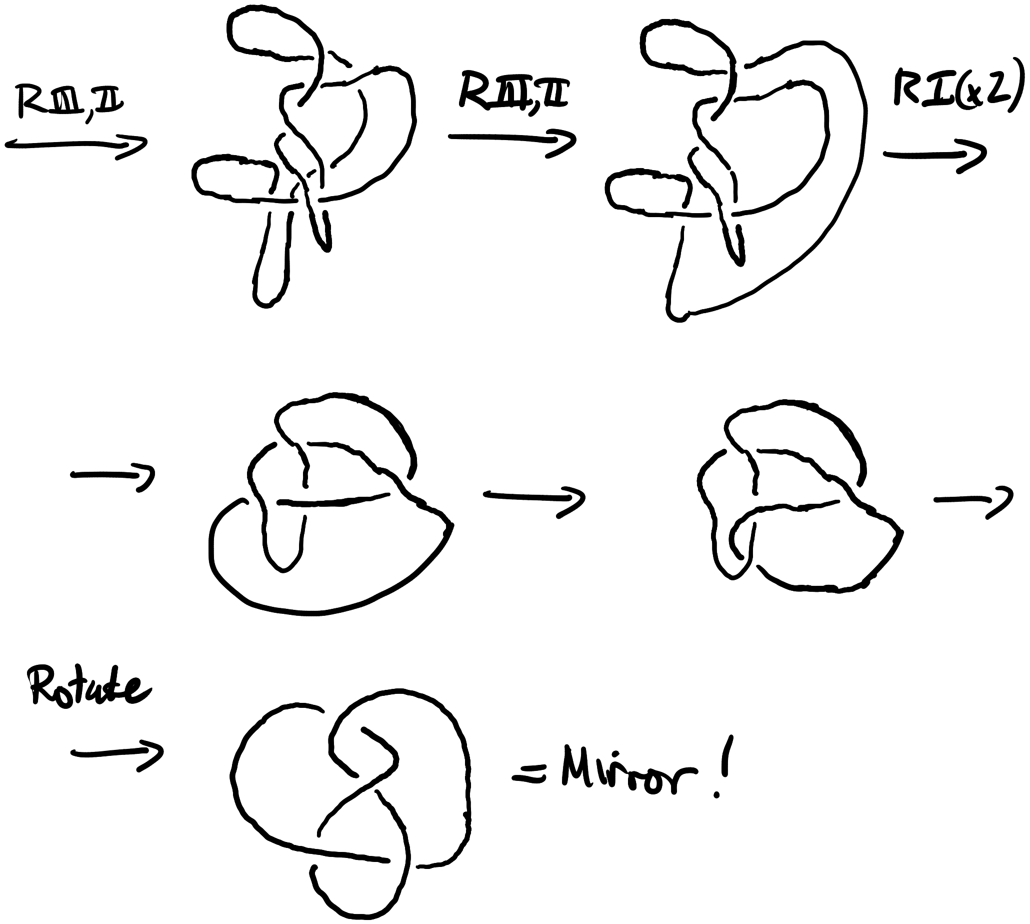


Proposition: These knots are equivalent.

Proof:



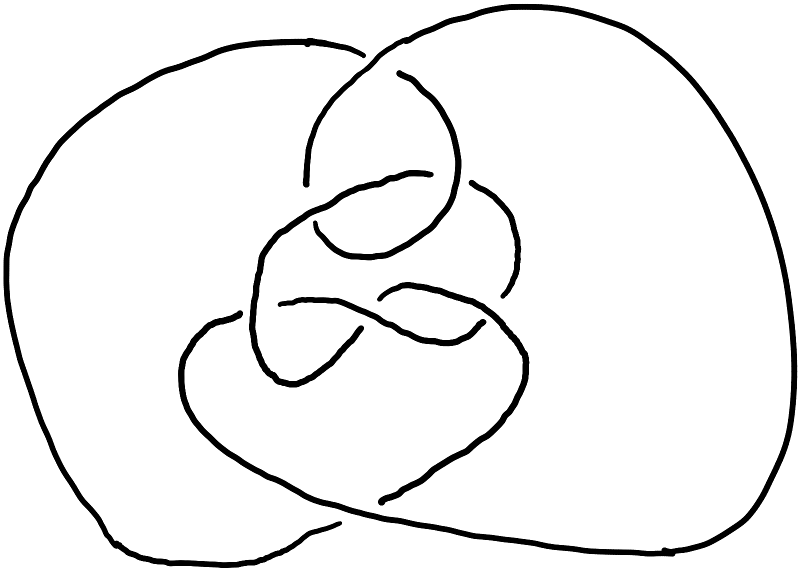



RIII,II
RIII,II
RI(x2)
Rotate
=Mirror!

YSP Knots

July 3, 2019

FACT:
This is an unknot.



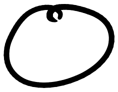
Problem 1: Find a sequence of Reidemeister moves that take this to the unknot.

Problem 2: Show that, in any solution to problem 1, there will be intermediate diagrams with <u>more</u> crossings than the seven that appear in the original diagram.

Hint: You can solve problem 2 without solving problem 1.

YSP Knots July 8, 2019

Two knot diagrams represent the same knot if and only if they can be connected by Reidemeister moves.

Problem: How to actually prove  and  are not equivalent? How to prove two given knots that "look" different are not equivalent?

What's the issue? Two knot diagrams could be equivalent, but the sequence of moves that connect them could be so long that we can't find it.

Need to have tools that can show that two knot diagrams cannot be connected by moves.

The concept of a "knot invariant": We want to attach a quantity (numerical, boolean, vector, ...) to each knot diagram. This quantity is called invariant if it always takes the same value on equivalent diagrams.

If we call the invariant I, this amounts to saying: for knot diagrams D_1 and D_2, if D_1 and D_2 are equivalent, then $I(D_1) = I(D_2)$

[But $I(D_1) = I(D_2)$ need not imply D_1 equivalent to D_2]

Since equivalent knot diagrams are related by Reidemeister moves, it is enough to show that the value of the invariant does not change when a move is applied.

Some knot invariants are easy to define.

Definition The crossing number of a knot K is the minimal number of crossings that can appear in any diagram for K. We denote it $c(K)$.

Problem Why is this a knot invariant?

However, $c(K)$ is basically impossible to compute directly, since its definition involves considering all of the infinitely many diagrams for K.

Q: Why don't we just count the crossings for some diagram for K?

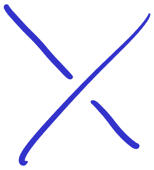
Coming up with a computable knot invariant requires some cleverness. Here is an example

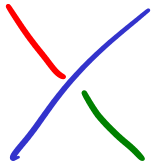
Tricolorability: This is a True/False (Boolean) property of a diagram:

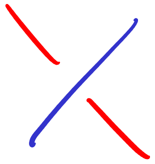
$$\text{Tricolorable?} : \left\{ \begin{matrix} \text{knot/link} \\ \text{diagrams} \end{matrix} \right\} \longrightarrow \{T, F\}$$

To define it: Take a knot/link diagram D. At each crossing, we will for now regard the understrand as broken into two pieces.

Then, we color the strands with 3 colors. At each crossing, there are 3 strands (overstrand and two parts of understrand.) The coloring is a *valid* tricoloring if, at each crossing we see either all one color or all 3 colors


valid

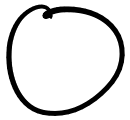

valid

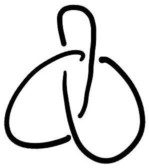

not valid.

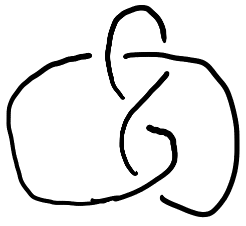
A tricoloring is *trivial* if all strands have the same color, and *nontrivial* if at least two colors are used.

Definition: A diagram is *tricolorable* if it can be given a nontrivial valid tricoloring.

Eg


not tricolorable


tricolorable

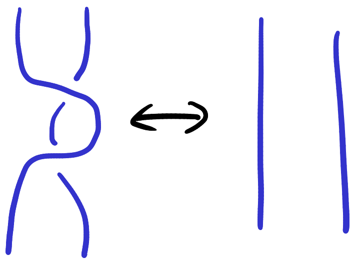

Not tricolorable.

Theorem: Tricolorability is a knot invariant.

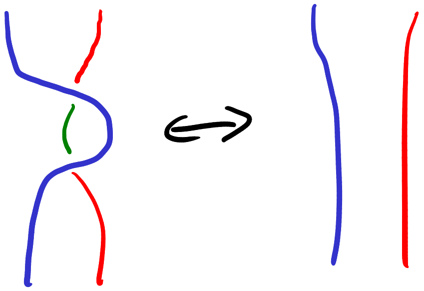
Proof: Need to show tricolorability is preserved under Reidemeister moves.

Eg:

RII



or

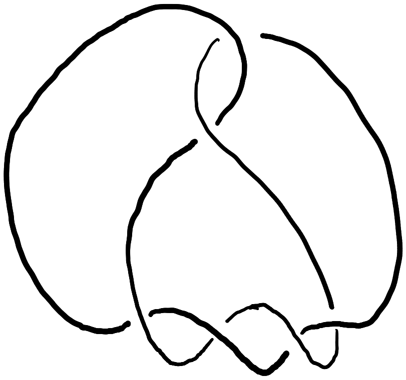


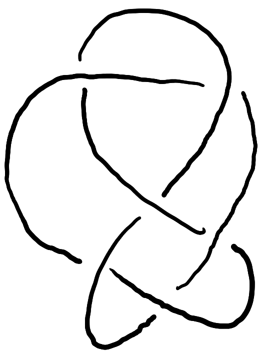
Problems:

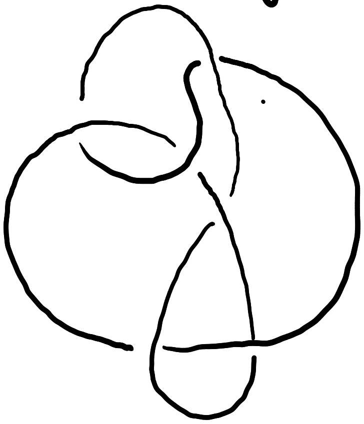
1. Complete the proof by analysing the effect Reidemeister moves I and III.

2. Determine the tricolorability of the knots/Links in last weeks hand outs. Which knots/Links does it distinguish?

3. Determine tricolorability for the following knots







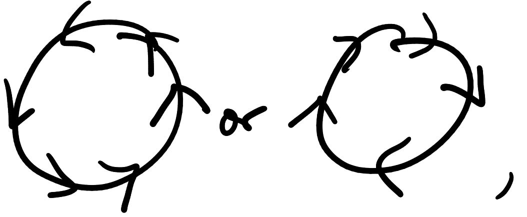
4. Construct an infinite sequence of tricolorable knots (you need not prove they are actually distinct)

*5. Can you generalize tricolorability to more than 3 colors?

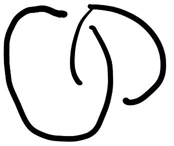
YSP Knots July 9, 2019

Linking number This is a rather simple invariant that helps us see whether two loops are linked. It depends on a choice of orientation for the components of the link.

Given a loop, an orientation is a choice of direction along the loop.



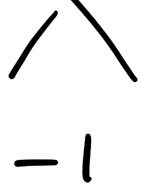
A loop has two possible orientations. Switching from one to the other is called reversing the orientation.

Now consider a link L with 2 components (2 loops) such as .

1. Choose an orientation for each component.
2. Look for crossings where the two components cross. Ignore crossings that a component makes with itself.
3. At each such crossing, identify it as either

+1

or



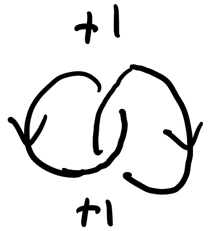
−1

4. Add up these ±1 numbers, and divide the sum by 2.

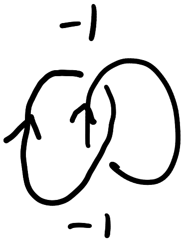
The result is called the linking number of the two components.

Eg

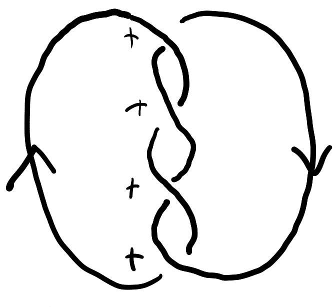
Linking # = 0



linking # = 1



Linking # = -1



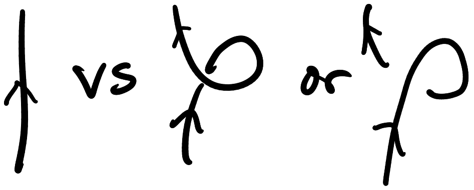
linking # = 2

<u>Problem</u>: Why is the linking number always an integer? (we divided by 2...)

<u>Problem</u>: What happens to the linking number if reverse the orientation of one component? of both components?

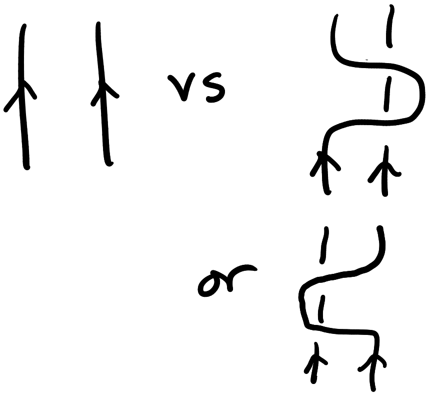
<u>Theorem</u> Linking number is an invariant. It does not change when Reidemeister moves are applied.

<u>Proof</u> I

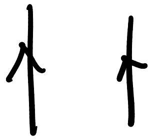


Never contributes to linking # since only one component is involved

II

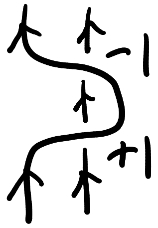


If two strands belong to same component, neither contributes two strands belong to different components:



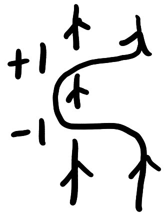
contributes 0

and



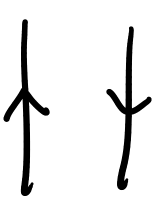
contributes 0

Also

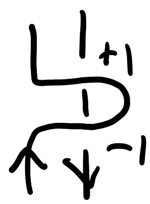


contributes 0.

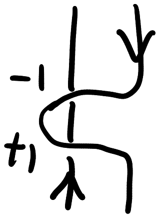
There are also the cases



vs

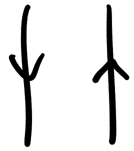


or

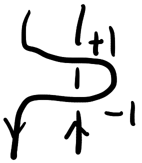


all contribute 0.

and



vs

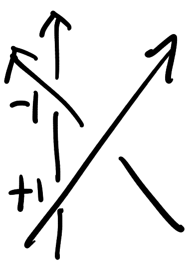


or

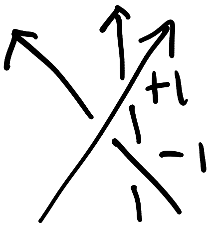
all contribute 0.

III: There are 3 strands: two will be from one component, and the third from other component.

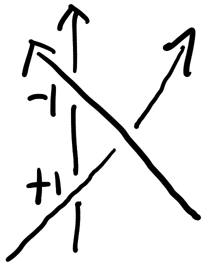
Case: bottom strand is from other component:



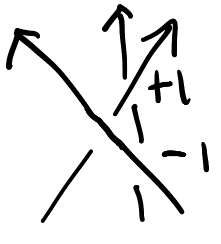
vs



or



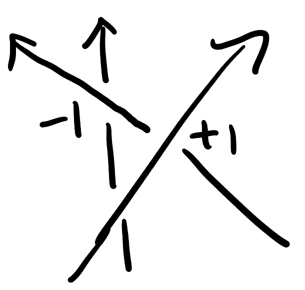
vs



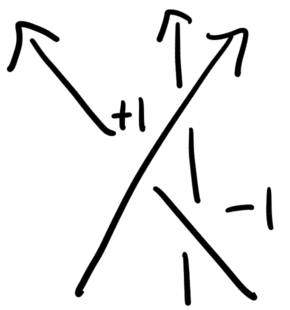
Always contributes 0. Still does so if we reverse any orientations.

Case where "top" strand is from other component works similarly

Case where "middle strand is from other component.



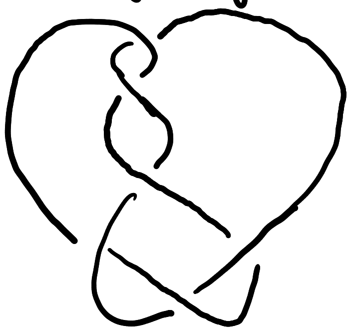
vs

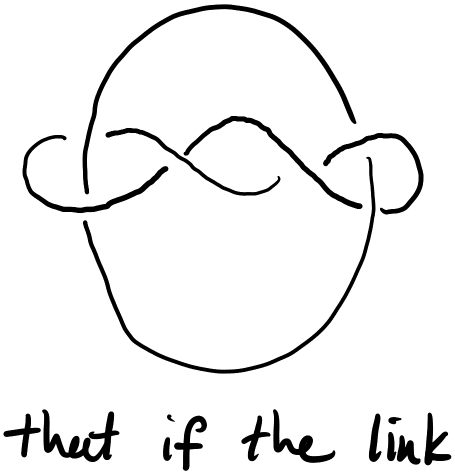


etc.

Shorter argument for III: Each diagram has one crossing between the pairs top/middle, middle/bottom, top/bottom, and the sign of that crossing does not change when we do the move.

Problems 1. Compute linking number for the following links (using some orientations)



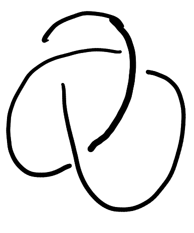


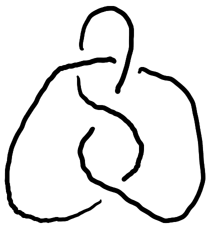
2. Is it necessarily true that if the linking number is zero then the link is equivalent to the unlink ○○ ?

3. What happens if you count *all* crossings in a knot or link diagram with the signs used for linking number? Is it an invariant?

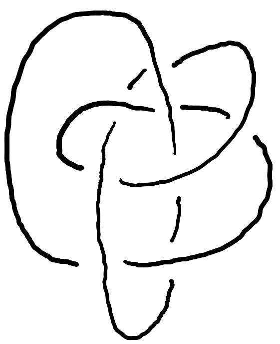
YSP Knots July 10, 2019

Alternating knot diagrams. You may have noticed that many diagrams have the property that as you move along a strand, it alternates going over and under. Such a diagram is called *alternating*. A knot or link is called alternating if it is possible to draw an alternating diagram for it. Not all knots are alternating.



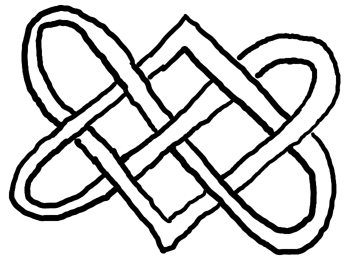


Alternating.



Not alternating. This knot has no alternating diagram.

Alternating knots are aesthetically pleasing. Many knots in artworks are alternating. See "Celtic knots"

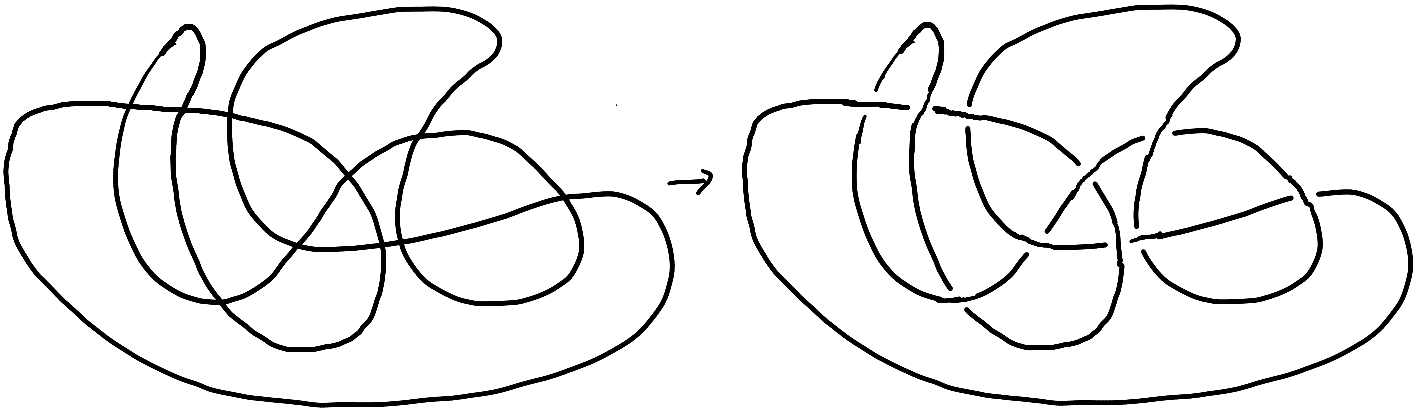


We shall see that alternating knots have interesting mathematical properties as well.

Constructing an alternating knot.

1. Draw a curve in the plane that crosses itself several times.

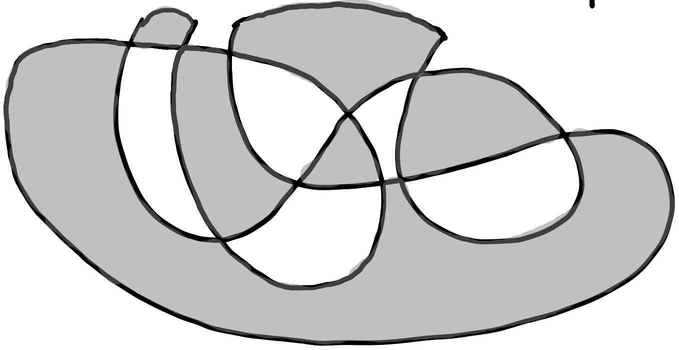
2. Pick a crossing, and choose which strand is over. Then go along curve and try to make it alternate.



Question: Does this always work? Do we ever run into a situation where we cannot proceed because something that is supposed to be over was already chosen to be under?

Here is another question that turns out to be related. Suppose we "flatten" a knot diagram to get a self-intersecting curve in the plane.

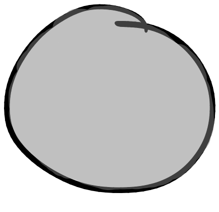
Now, we want to shade the regions in the complement of the curve in a "checkerboard" pattern.



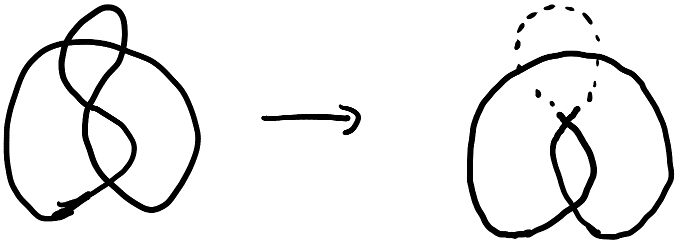
Question: Why is this always possible?

One argument uses induction on the number of crossings.

Base case: Zero crossings
This works
(Jordan curve theorem)



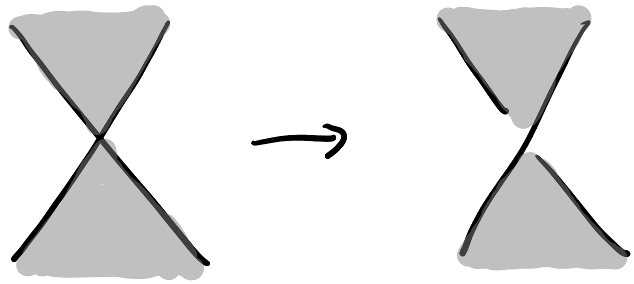
Induction step: Starting with a selfintersecting curve, find a piece of it that forms a non self-intersecting closed loop, and remove that loop



The result has fewer crossings, so by induction it admits a checkerboard coloring

Now, add back the loop we deleted, and flip the color of everything inside this loop. The result is a checkerboard coloring of the original curve.

Now given a checkerboard coloring, we can introduce over/under strands according to the rule



<u>Problem</u>: Show that this rule always results in an alternating diagram, thus resolving the question from page 2.

<u>More problems</u>:

1.a. Suppose we have an alternating diagram, and we want to apply a single Reidemeister move to it. Which moves are possible?

b. Suppose we want to apply a Reidemeister move to an alternating diagram so that it remains alternating. Which moves are possible?

2. Suppose a (not necessarily alternating) knot diagram is given with n crossings. What is the maximum number of crossings that may need to be changed in order to make the diagram alternating?

3. Show that by changing some crossings, any knot may be turned into the unknot.

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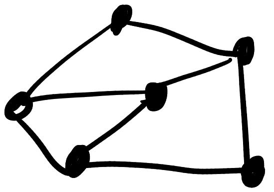
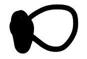
The existence of the checkerboard coloring for a knot diagram can be reformulated.

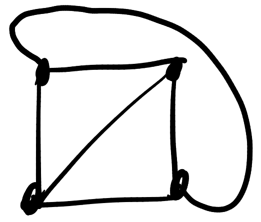
<u>Definition</u> A <u>graph</u> consists of a set of vertices and a set of edges connecting them.



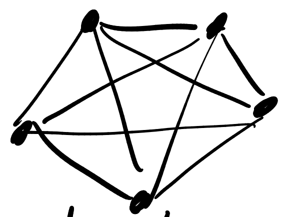
We allow  multiple edges

 loops

A <u>planar graph</u> is a graph that is embedded in the plane so that edges never intersect (except at the vertices of course)



planar

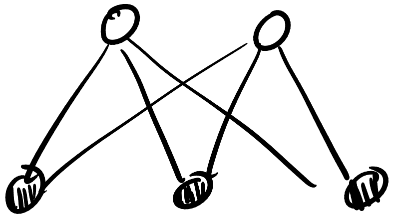


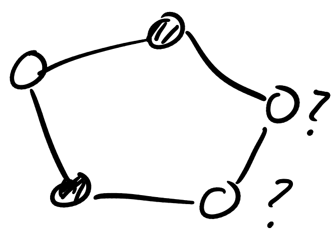
not planar

Vertices are <u>adjacent</u> if they are connected by an edge.

A graph is <u>bipartitie</u> if we can color the vertices with two colors so that adjacent vertices have different colors

Bipartite

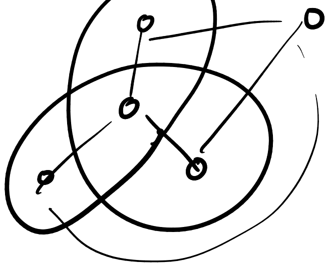




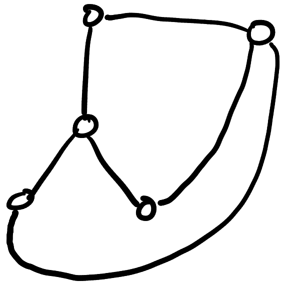
Not bipartite

Given a knot diagram we can construct a graph called the <u>dual graph</u>: Vertices are regions in the complement of the diagram, and we draw an edge if two regions are adjacent. Remember to include the region outside the knot diagram

Trefoil



Dual graph



Knot diagram admits checkerboard coloring $\Longleftrightarrow$ dual graph is bipartite.

Since we know any knot diagram admits a checkerboard coloring, the dual graph is always bipartite.
The dual graph is also obviously planar.

<u>Question</u> Is every bipartite planar graph the dual graph of some knot or link diagram?

There is another graph we can associate to knot diagram, called the <u>medial graph</u>. Its vertices are a subset of the vertices of the dual graph, but its edges correspond to crossings.

July 11, 2019

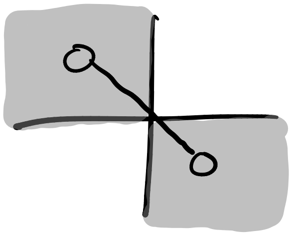
To construct it:

1. Construct checkerboard shading of knot diagram.

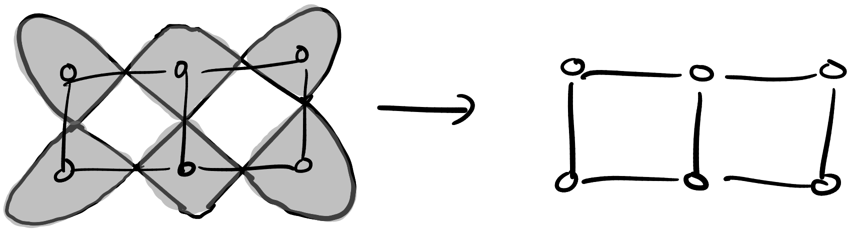
2. Place a vertex in each shaded region.

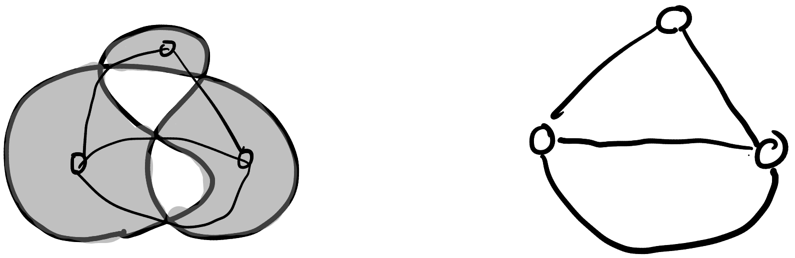
3. Place edge between vertices if the shaded regions are diagonally opposite at a crossing:

i.e. locally

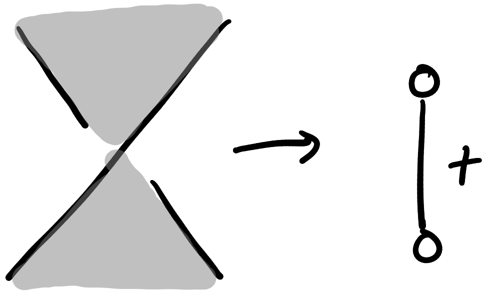


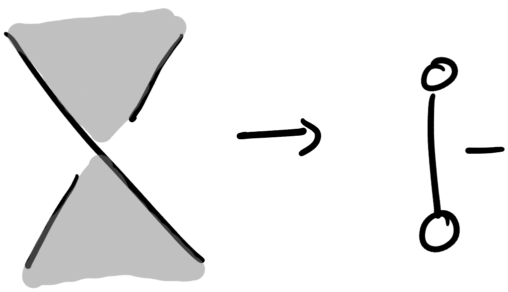
E.g.





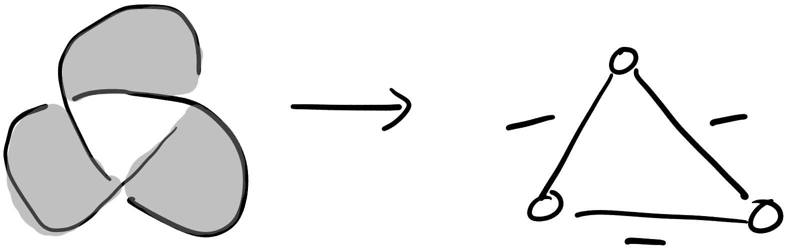
Note that edges correspond to crossings.
So far we have ignored the over/under strands at crossings. We can remember this by putting a +/− on each edge according to the rule





Note: This is similar to, but independent from the +/− convention for linking number.

e.g.

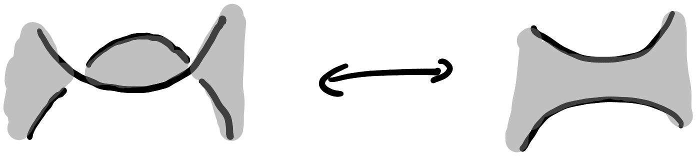


The result is called a signed planar graph, the medial graph of the knot diagram.

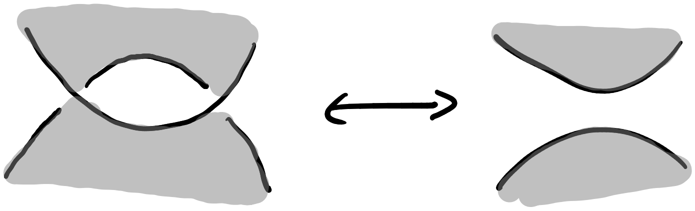
Question: Is every signed planar graph the medial graph of some knot or link diagram?
Can you reconstruct the knot or link from its medial graph

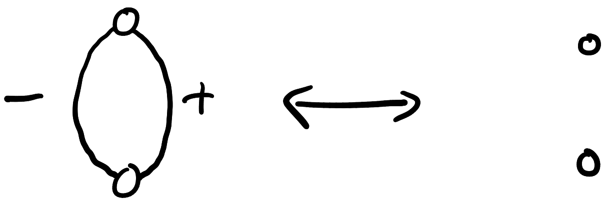
Problem: Work out the effect of Reimeister moves on the medial graph. There are several cases for each move.

Eg II:



or





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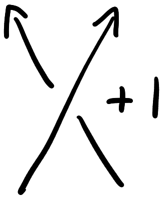
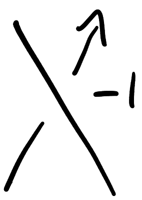
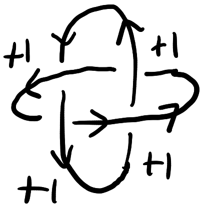
Recall linking #:





eg



total = 4 linking # = 4/2 = 2

Previous proplem: What happens if we count <u>all</u> crossings in an oriented knot or link diagram?

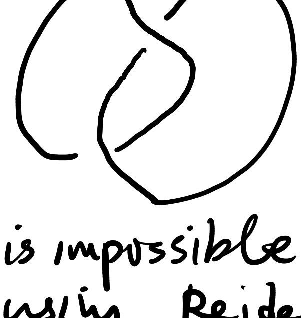
total = 3

total = 0

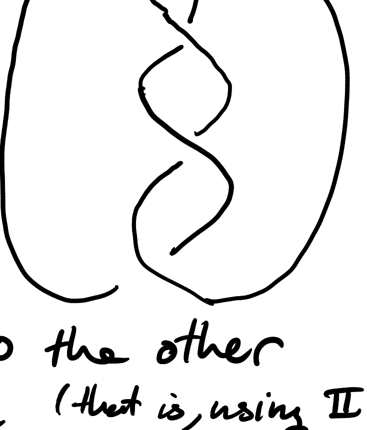
This total is called the <u>writhe</u> (or Tait number) of the diagram.

<u>Question</u>: Is the writhe a link invariant? If we take the writhe of a knot diagram, does it depend on the choice of orientation.

<u>Problem</u>:



is equivalent to



Show it is impossible to get from one to the other without using Reidemeister I move (that is, using II + III only)

The only real knot invariant we have so far is tricolorability. This can't distinguish unknot and figure 8 knot.

Hence we look for better knot invariants. A pretty good one takes values in <u>polynomials</u>.

A <u>polynomial</u> is an expression in several variables A, B, C, ... (a.k.a. indeterminates) like

$$3A^2C + 6BC + 4A^5 + 22A^6B^5C^3$$

i.e., any expression that can be obtained starting with constants, A, B, C, and combining them using addition, subtraction, and multiplication. (not division) All usual rules of algebra that hold for numbers are assumed to be valid for A, B, C as well.

e.g. $(5A+C)B = 5AB+CB$ distributive law

$AB=BA$ commutative law,

$0+A=A$ $A-A=0$

$(AB)C = A(BC)$. Full list: "axioms of commutative ring with unit"

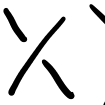
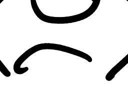
If you like, you can think of A, B, C as representing unknown numbers, but we are <u>not</u> (necessarily) trying to solve for them.

July 11, 2019

The Kaufmann bracket

[Louis Kaufmann (UIC)
Building on Jones 1985]

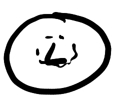
<u>Theorem</u>: There is a unique way to associate a polynomial in A, B, C to any Link diagram, notated $L \mapsto \langle L \rangle$

Such that (i) $\Big\langle$  $\Big\rangle = A\langle\,)(\,\rangle + B\Big\langle$  $\Big\rangle$

Any crossing

(ii) $\langle L \quad o \rangle = C\langle L \rangle$

link + disjoint circle

Also $\Big\langle$  $\Big\rangle = C\langle L \rangle$

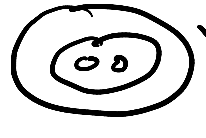
and (iii) $\langle o \rangle = 1$

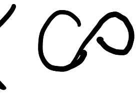
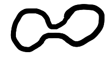
unknot

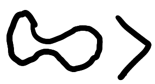
This is a recursive definition: the idea is to use rule (i) to reduce the number of crossings, until there are none, and then to use (ii) and (iii) to finish up.

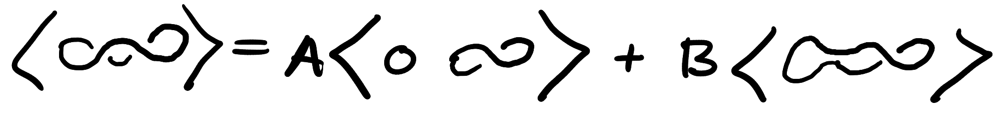
Eg $\langle o \rangle = 1 \qquad \langle o \ o \rangle = C \qquad \langle o\ o\ o \rangle = C^2$

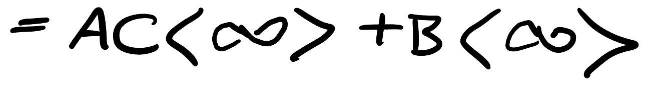
$$\langle \underbrace{o \cdots o}_{\text{n disjoint circles}} \rangle = C^{n-1}$$

$\Big\langle$  $\Big\rangle = C^3$

$\Big\langle$  $\Big\rangle = A\langle o\, o \rangle + B\Big\langle$  $\Big\rangle = AC + B$

$\Big\langle$  $\Big\rangle = A\Big\langle$  $\Big\rangle + B\langle o\, o \rangle = A + BC$



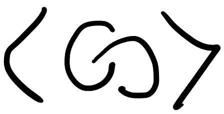


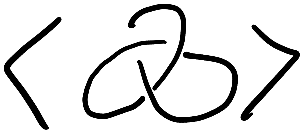
$$=(AC+B)(A+BC) = A^2C + ABC^2 + AB + B^2C$$

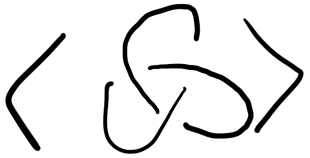
As you can see, this isn't a knot invariant...
But we will press on! We make it a knot invariant by choosing particular values/relationships between the variables.

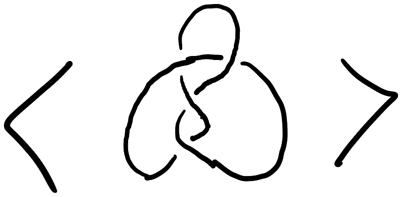
Next time: Prove existence and uniqueness, analyse question of achieving invariance.

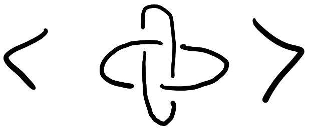
Problems: 1. compute



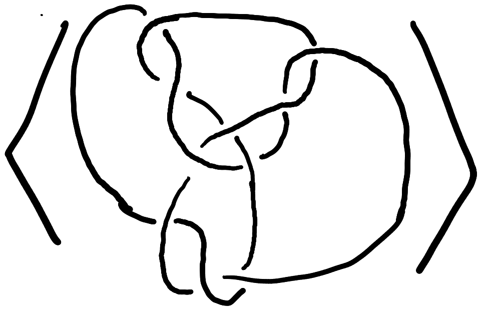








2. How long is it going to take to compute
Approximately how many terms are there?



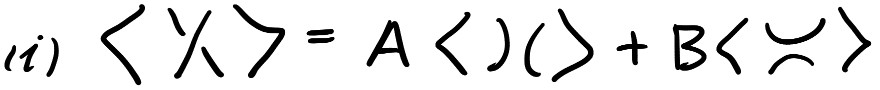
?

3. If $L^!$ represents the mirror of L (all crossings reversed) what relationship is there between

$\langle L \rangle$ and $\langle L^! \rangle$?

YSP Knots July 15, 2019

Last time we introduced the Kaufmann bracket polynomial
It satisfies the rules

(i) 

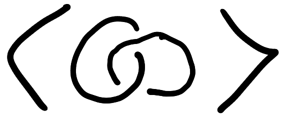
(ii) $\langle O \boxed{L} \rangle = C \langle L \rangle$

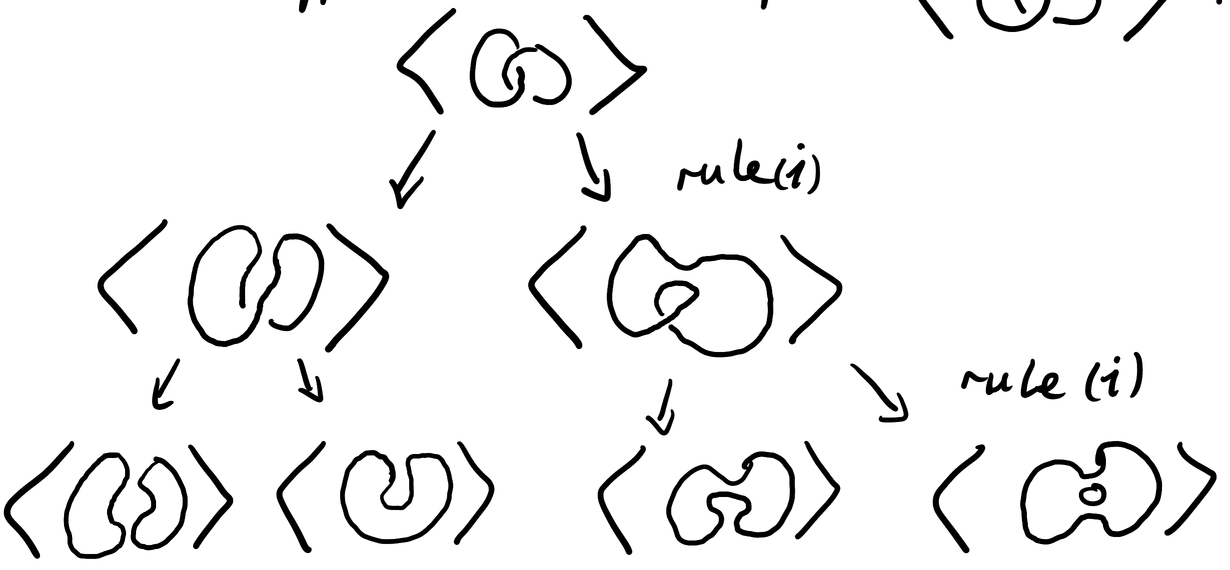
(iii) $\langle O \rangle = 1$

We proved last time that these rules completely determine $\langle L \rangle$ for any link diagram L.

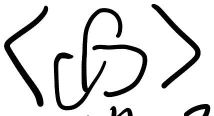
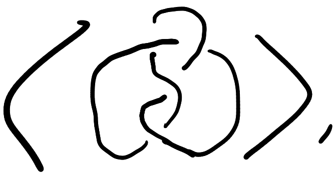
However, we need to show these rules are actually consistent, that is, we need to show existence of a rule $L \mapsto \langle L \rangle$ that actually satisfies (i), (ii), (iii).

We can do this by finding a non-recursive formula for $\langle L \rangle$

Q: What happens when we compute  ?



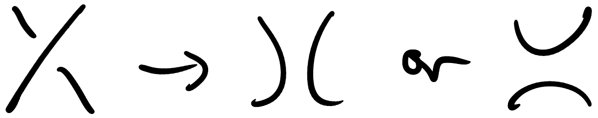
4 terms

Q: If we do  or , how many diagrams do we end up with?

A: 8, 16, general 2^n where $n = \#$ of crossings.

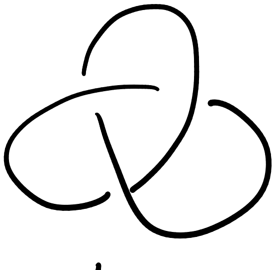
Q: Can we predict what the final diagrams will look like with out doing the steps?

For each of the n crossings, we are going to need to consider Both ways of splitting it

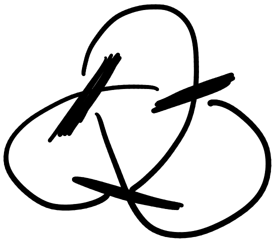


Since we have 2 choices at each crossing, we have 2^n overall.

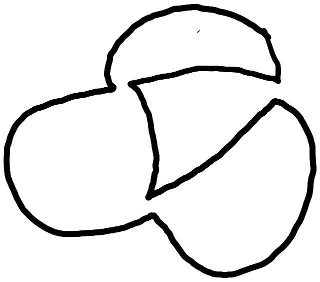
For a link diagram L, let us call a "state" of L a choice of which way to split at each vertex



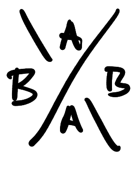
L



A state of L



Result of splitting by this state

When we apply rule (1), we multiply by certain powers of A, B. Can we predict these? Lets mark the quadrants at each crossing as 

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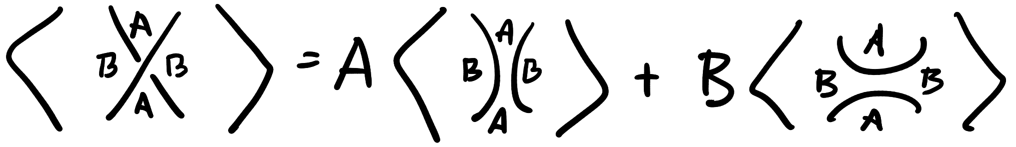
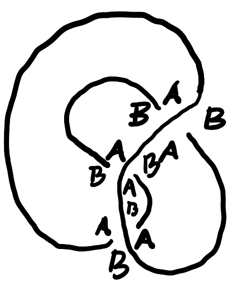
So rule (1) looks like



A's get connected

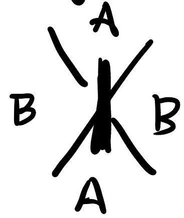
B's get connected

Note: regions can have multiple labels eg

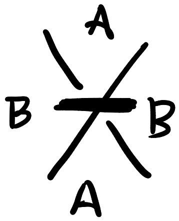


So each ~~state~~ picks up a power of A whenever one of the splits it contains joins two A quadrants, and a power of B whenever a split joins two B quadrants.

Def:



=: "A-split",



=: "B-split".

Def: If S is a state, set

$\alpha(S) :=$ number of A-splits in S.

$\beta(S) :=$ number of B-splits in S.

Once we perform all of the splits, we are left with a bunch of disjoint circles.

Def: $\gamma(S) :=$ number of circles obtained after splitting by state S.

$$\langle L \text{ split by } S \rangle = \langle \gamma(S) \text{ circles} \rangle = C^{\gamma(S)-1}$$

Putting it all together, we obtain a formula for $\langle L \rangle$

$$\langle L \rangle = \sum_{\substack{\text{states } S \\ \text{of } L}} A^{\alpha(S)} B^{\beta(S)} \langle L \text{ split by } S \rangle \qquad (\text{rule } i)$$

$$= \sum_{\substack{\text{states } S \\ \text{of } L}} A^{\alpha(S)} B^{\beta(S)} C^{\gamma(S)-1} \qquad (\text{rules } ii + iii)$$

The sum is over all 2^n states of L (n = number of crossings in L)
This is the <u>state sum formula for the Kaufmann bracket.</u>

If we were to <u>define</u> $\langle L \rangle$ by this formula, we get something that satisfies rules (i), (ii), (iii). This proves the existence of something satisfying the rules. We already showed it's unique.

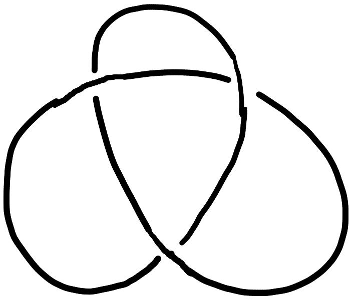
Proof of rule (i) from state sum formula: Let L be a link diagram, and pick a crossing x. Set $L_A = L$ with A-split at x
$L_B = L$ with B-split at x.
Use subscripts α_L α_{L_A} α_{L_B} to keep track of which diagram we are talking about.
Need

$$\sum_{\substack{\text{states} \\ S \text{ of } L}} A^{\alpha_L(S)} B^{\beta_L(S)} C^{\gamma_L(S)-1} = A\left(\sum_{\substack{\text{states} \\ S_A \text{ of } L_A}} A^{\alpha_{L_A}(S_A)} B^{\beta_{L_A}(S_A)} C^{\gamma_{L_A}(S_A)-1} \right)$$
$$+ B\left(\sum_{\substack{\text{states} \\ S_B \text{ of } L_B}} A^{\alpha_{L_B}(S_B)} B^{\beta_{L_B}(S_B)} C^{\gamma_{L_B}(S_B)-1} \right)$$

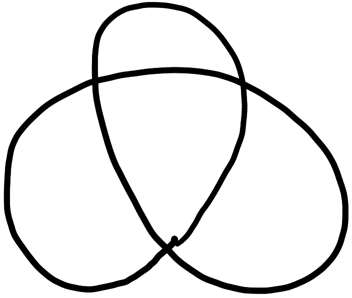
Each state S_A of L_A corresponds to a state S of L with an A split at x, and $\alpha_{L_A}(S_A)+1 = \alpha_L(S)$ $\quad \beta_{L_A}(S_A) = \beta_L(S)$ $\quad \gamma_{L_A}(S_A) = \gamma_L(S)$.
Similarly for states of L_B, so terms match. [Give details]

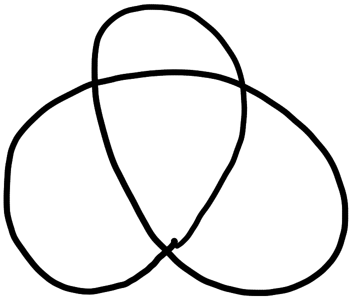
States Worksheet for Trefoil

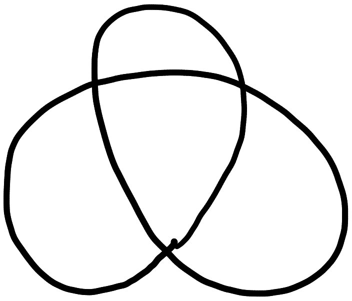
Label A, B quadrants :

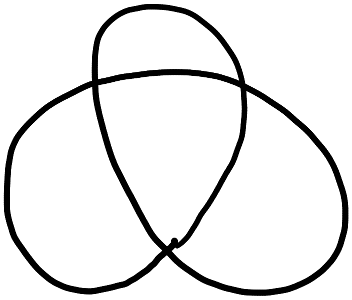


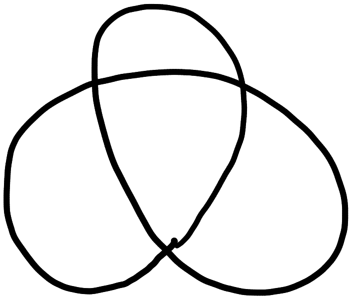
States :

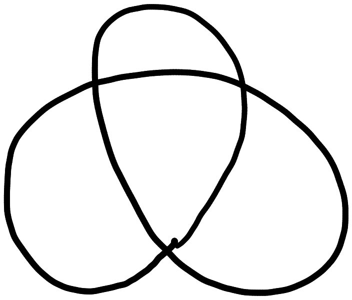


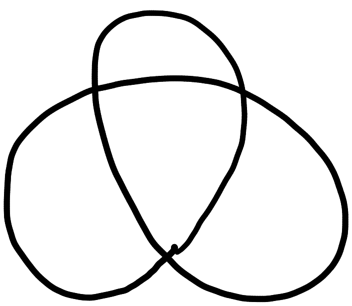


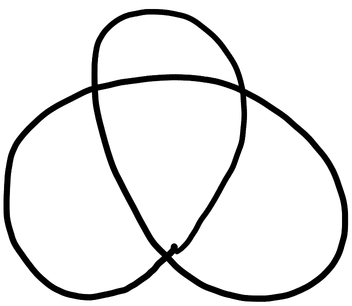






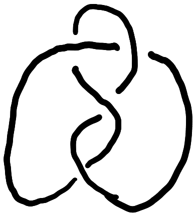


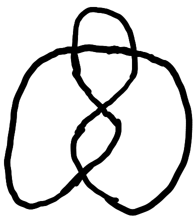


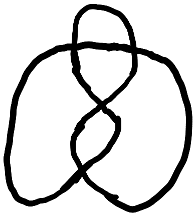


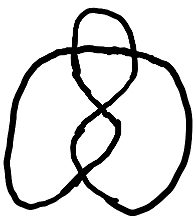
Total =

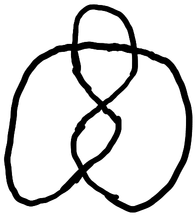
States worksheet for figure 8 knot

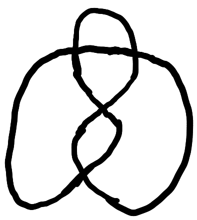


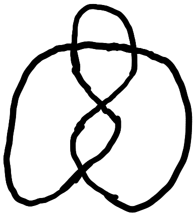


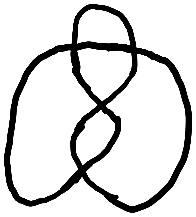


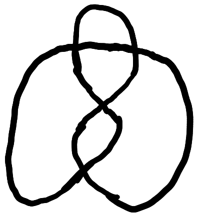


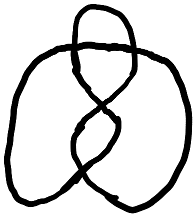


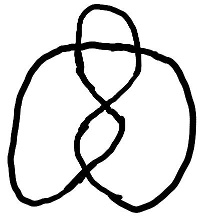


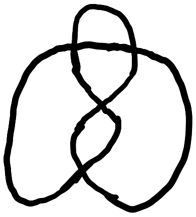


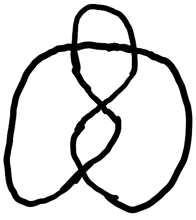


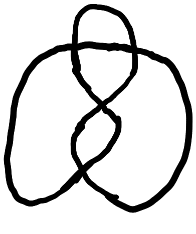


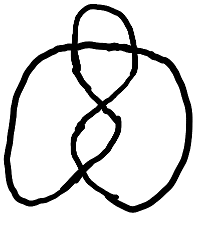


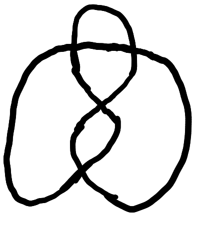


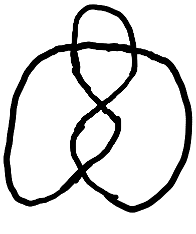








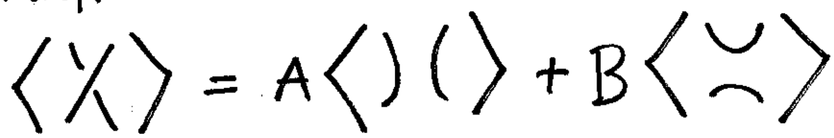




YSP Knots

July 16, 2019

Kaufmann bracket

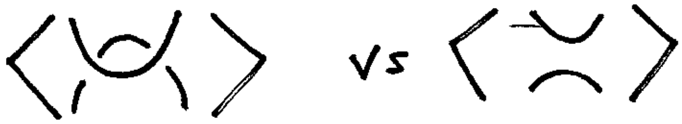
(i) 

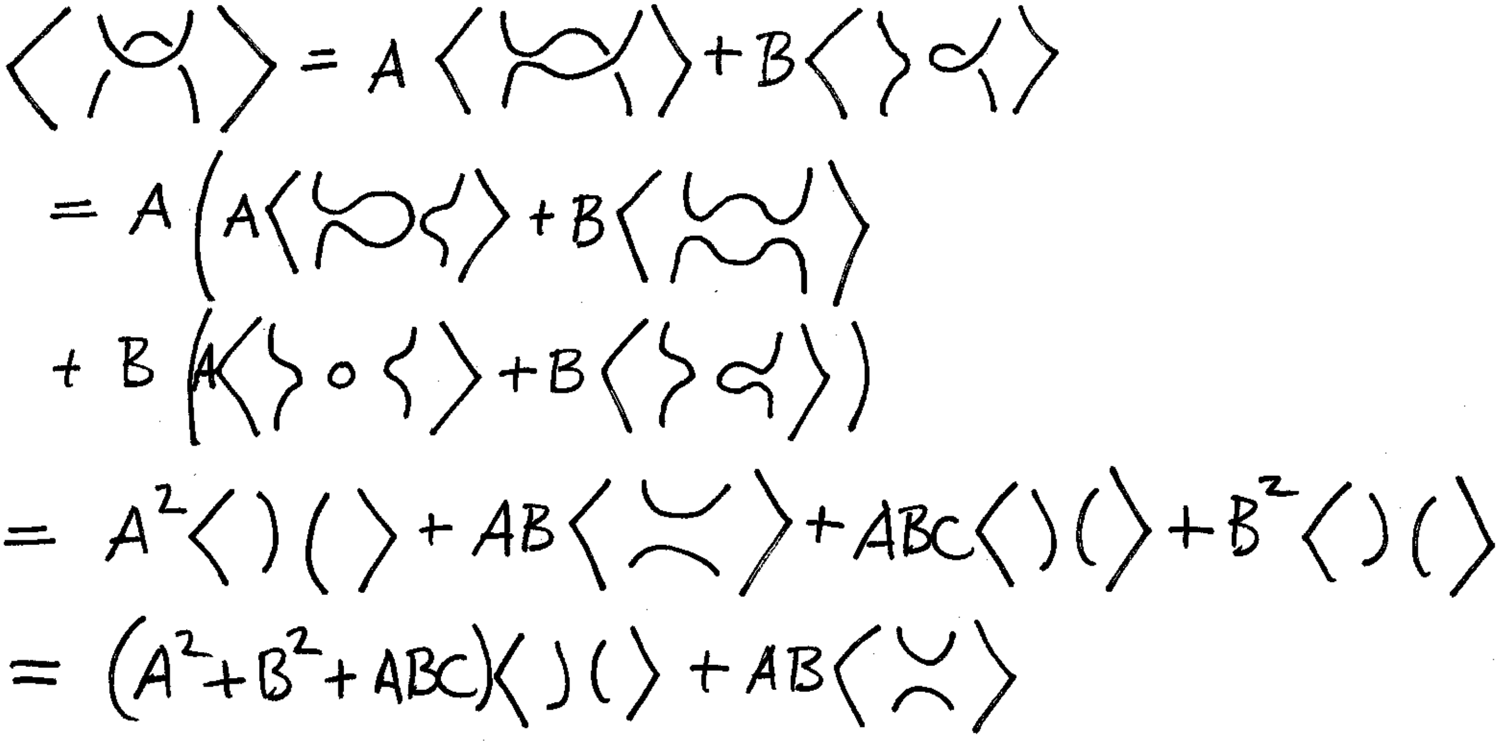
(ii) $\langle \bigcirc \boxed{L} \rangle = C \langle L \rangle$

(iii) $\langle \bigcirc \rangle = 1$

We already saw that $\langle L \rangle$ is not invariant under Reidemeister moves. This [redacted] was by design; we now seek conditions on A, B, C that gaurantee invariance.

Let us start with move II:





in order for this to be equal to $\langle \asymp \rangle$,

we need the equations $\left\{ \begin{array}{l} A^2+B^2+ABC=0 \\ AB=1 \end{array} \right\}$ to be true.

Since we invented the variables A, B, C, we are free to impose these conditions on the variables.

<u>Note</u>: $AB=1$ iff $B=A^{-1}$

If $B=A^{-1}$ then the other equation becomes

$$A^2+A^{-2}+C=0 \quad \text{or} \quad C=-A^2-A^{-2}.$$

Let us define a modification of the Kaufmann Bracket that takes this into account.

<u>Definition</u>: $\langle\!\langle L \rangle\!\rangle$ (Double bracket) is $\langle L \rangle$ but with B replaced by A^{-1} and C replaced by $-A^2-A^{-2}$

It satisfies

(i) $\langle\!\langle \times \rangle\!\rangle = A \langle\!\langle)\,(\rangle\!\rangle + A^{-1} \langle\!\langle \asymp \rangle\!\rangle$

(ii) $\langle\!\langle \bigcirc \boxed{L} \rangle\!\rangle = (-A^2-A^{-2}) \langle\!\langle L \rangle\!\rangle$

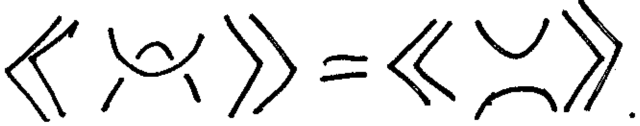
(iii) $\langle\!\langle \bigcirc \rangle\!\rangle = 1$

(State sum formula)

$$\langle\!\langle L \rangle\!\rangle = \sum_{\substack{\text{states } s \\ \text{of } L}} A^{\alpha(s)} A^{-\beta(s)} (-A^2-A^{-2})^{\gamma(s)-1}$$

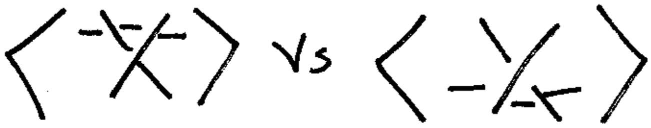
July 16, 2019

Proposition ⟪L⟫ is invariant under Reidemeister II move.



Proof: Recite the steps that led us to the equations defining ⟪L⟫

Next: Reidemeister III. If we do a similar analysis of

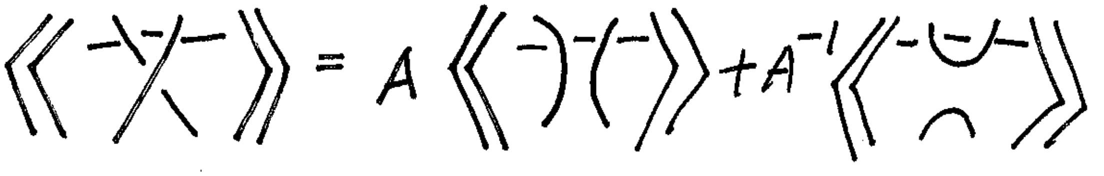


we get a bunch of equations that A, B, C must satisfy in order for these two things to be equal. The miracle is that these equations are already implied by $AB = 1$ and $A^2 + B^2 + ABC = 0$

That is:

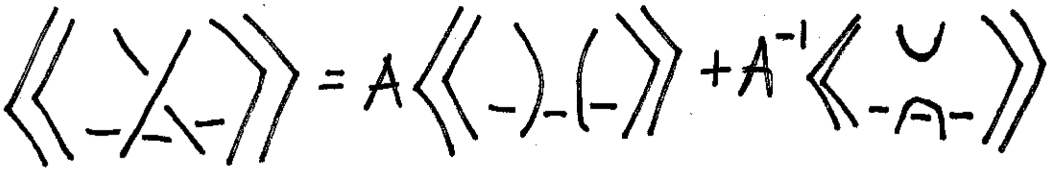
Proposition: ⟪L⟫ is invariant under Reidemeister III move

Proof:

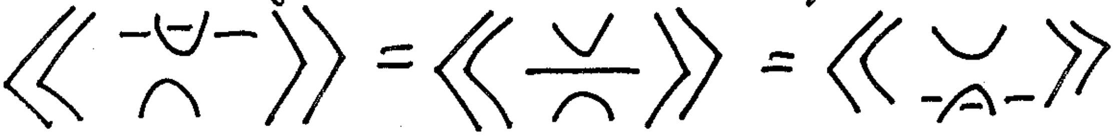


(split "center" crossing)

vs

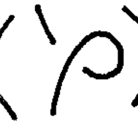
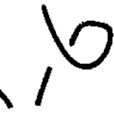


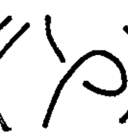
The two right hand sides are equal since



by invariance under move II.

Last: Move I: we use $\langle\langle L \rangle\rangle$

$\langle\langle\,) \,\rangle\rangle$ vs $\langle\langle$  $\rangle\rangle$ or $\langle\langle$  $\rangle\rangle$

$\langle\langle$  $\rangle\rangle = A \langle\langle$  $\rangle\rangle + A^{-1} \langle\langle$  $\rangle\rangle = \left(A\left(-A^2 - A^{-2}\right) + A^{-1}\right) \langle\langle\,) \,\rangle\rangle$

$= -A^3 \langle\langle\,) \,\rangle\rangle$

Exercise: $\langle\langle$  $\rangle\rangle = -A^{-3} \langle\langle\,) \,\rangle\rangle$

So we could achieve invariance if we set $-A^3 = 1$ ie. $A^3 = -1$. This implies A has one of the values

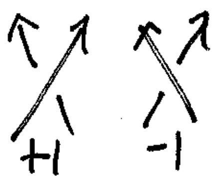
~~$A = -1$~~, $e^{\pi i/3} = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$,

$e^{-\pi i/3} = \cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

Proposition: $\langle\langle L \rangle\rangle$ is a link invariant if we set $A = -1$, $e^{\pi i/3}$, or $e^{-\pi i/3}$

Problem: evaluate these invariants for some links/knots. See any patterns?

Problem: Recall writhe of an oriented link writhe(L) = count of crossings according to



Prove that writhe is invariant under moves II and III, and that

writhe() = writhe( )) − 1

writhe() = writhe( )) + 1

(any orientation)

YSP Knots

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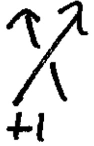
$\langle\langle L\rangle\rangle$ defined by

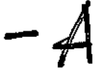
(i) $\langle\langle$  $\rangle\rangle = A\langle\langle)(\rangle\rangle + A^{-1}\langle\langle$  $\rangle\rangle$

(ii) $\langle\langle O\ L\rangle\rangle = (-A^{2}-A^{-2})\langle\langle L\rangle\rangle$

(iii) $\langle\langle O\rangle\rangle = 1$

If L is an oriented link, define writhe $w(L)$ to be count of crossings  . with signs.

Both $\langle\langle L\rangle\rangle$ and $w(L)$ are invariant under moves II and III, but not move I. In fact:

$\langle\langle$  $\rangle\rangle = -A^{3}\langle\langle$  $\rangle\rangle$ $w($$) = w($$)+1$

$\langle\langle$  $\rangle\rangle = -A^{-3}\langle\langle$  $\rangle\rangle$ $w($$) = w($$)-1$

Consider the combination $X(L) = (-A^{3})^{-w(L)}\langle\langle L\rangle\rangle$

This is clearly invariant under II and III, but also I:

$X($$) = (-A^{3})^{-w(}$$^{)}\langle\langle$  $\rangle\rangle$

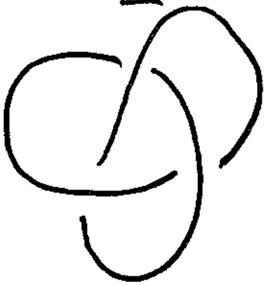
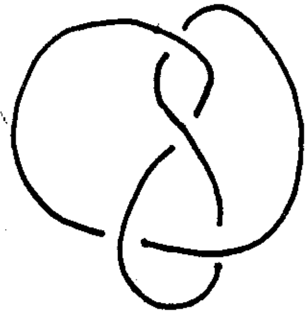
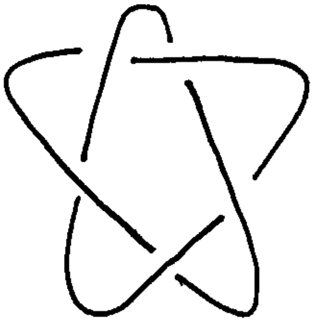
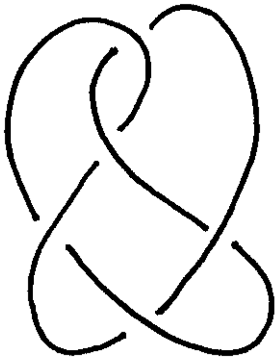
$= (-A^{3})^{-(w(}$$^{)+1)}(-A^{3})\langle\langle$  $\rangle\rangle = (-A^{3})^{-w(}$$^{)}\langle\langle$  $\rangle\rangle$

$= X($$)$

Note: The writhe of a link may depend on the choice of orientations, but the writhe of a knot doesn't.

so $X(K)$ does not depend on choice of orientation if K is a knot.

Here are some sample calculations. See if you can reproduce them.

L	X(L)
	$-A^{16}+A^{12}+A^{4}$
	$A^{8}-A^{4}+1-A^{-4}+A^{-8}$
	$-A^{28}+A^{24}-A^{20}+A^{16}+A^{8}$
	$-A^{24}+A^{20}-A^{16}+2A^{12}-A^{8}+A^{4}$

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We immediately conclude that these Knots are all distinct (non-equivalent) and distinct from the unknot $X(0) = 1$.

Question1 Are there two distinct (non-equivalent) Knots K_1 and K_2 with $X(K_1) = X(K_2)$?

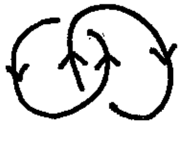
Question2 Is there a non-trivial knot K with $X(K) = 1$?

The answer to 1 is "yes". Can you find an example?
The answer to 2 is unknown.

Problem: If L is an oriented link, denote $L!$ the mirror of L, meaning L with all crossings reversed.

- Show that $X(L!)$ is $X(L)$ with A replaced by A^{-1}.
- If L is equivalent to $L!$, what can we conclude about $X(L)$?
- Show that  and  are not equivalent.

Notice that the exponents in the polynomials computed so far are always divisible by 4. Is this always true?

$X($  $) = -A^{-2} - A^{-10}$ not divisible by 4...

Problem: Show that for a knot (1 component) K all of the exponents in $X(K)$ are divisible by 4.

Easier problem: Show that for any oriented link L, all exponents of A in $X(L)$ are even.

Hint: state sum formula.

Problem Show that $X(L)$ satisfies the following relation: Suppose oriented links L_+, L_-, L_0 are the same except at one crossing, where they differ by

- Show that $A^4 X(L_+) - A^{-4} X(L_-) = (A^{-2} - A^2) X(L_0)$
- Show that this relation plus $X(\bigcirc) = 1$ is enough to compute $X(L)$ for any link L.

Definition The Jones polynomial of an oriented link L is $X(L)$ with A replaced by $t^{-1/4}$ We denote it $V(L)$. It is a polynomial with positive and negative powers of $t^{1/2} = \sqrt{t}$

It satisfies $V(\bigcirc) = 1$ and

$$t^{-1} V(L_+) - t V(L_-) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V(L_0).$$

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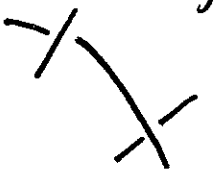
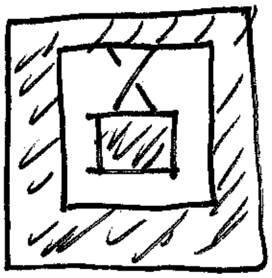
There are relationships between the exponents appearing in the Jones polynomial $X(L)$ or Kauffman bracket $\langle\langle L \rangle\rangle$ and the number of crossings in the diagram. The connection is closest for alternating diagrams. Alternating: over/under



Definition:
An alternating diagram is called simple if it does not contain an "isthmus":

or



This is the same as saying that, if we apply checkerboard shading, no region (shaded or unshaded) touches itself at a corner:

e.g.

e.g.

e.g.

Conjecture (Tait 1898) Any two simple alternating diagrams of the same link have the same number of crossings. This number is the minimal possible.

Theorem (Louis Kauffman, Kunio Murasugi, Morwen Thistlethwaite 1986)
Tait's conjecture is true.

The proof uses the polynomial $X(L)$ or $\langle\langle L \rangle\rangle$

Definition: If $P(A)$ is a Laurent polynomial in A,

maxdeg $P :=$ greatest exponent of A that appears in $P(A)$

min deg $P :=$ least exponent of A that appears in $P(A)$

span $P :=$ maxdeg $P -$ mindeg P

Example: $P = -A^{16} + A^{12} + A^4$ maxdeg $P = 16$ mindeg $P = 4$ span $P = 12$.

Example: $X(L) = (-A^3)^{-w(L)} \langle\langle L \rangle\rangle$, so

$$\text{maxdeg } X(L) = \text{maxdeg} \langle\langle L \rangle\rangle - 3w(L)$$

$$\text{mindeg } X(L) = \text{mindeg} \langle\langle L \rangle\rangle - 3w(L)$$

$$\text{span } X(L) = \text{span } \langle\langle L \rangle\rangle$$

Conclusion: span $\langle\langle L \rangle\rangle$ is a link invariant.

Theorem: If L is a simple alternating diagram, then

span $\langle\langle L \rangle\rangle = 4V$, where $V = \#$ crossings.

We need some preliminary lemmas:

Lemma: If $V = \#$ crossings and $R = \#$ regions in the diagram (including infinite region), then $R = V + 2$.

Proof: Thinking of a crossing as a vertex 

we can regard the diagram as a planar graph where every vertex has 4 edges coming out. let $E = \#$ edges.

July 18, 2019

Then $V-E+R=2$ (Euler's formula for planar graphs)

Also $E=2V$ (since 4-valent everywhere)

so $V-2V+R=2$ or $R=V+2$. ▨

<u>Lemma</u>: Let L be alternating. Mark quadrants near crossing as

. Choose a checkerboard shading. Then either all A's lie in shaded regions or all B's do.

"<u>Proof</u>":

<u>Main proposition</u>: Let L be simple and connected and alternating. Suppose diagram is shaded so that all A's are shaded (all B's unshaded). Then $\max\deg \langle\langle L\rangle\rangle = V+2(N-1)$ where N is the number of <u>unshaded</u> regions.

Also $\min\deg \langle\langle L\rangle\rangle = -V-2(M-1)$ where M is the number of shaded regions.

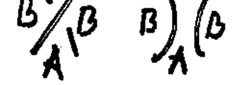
<u>Proof of theorem from main proposition</u>:

$$
\begin{aligned}
\text{Span} \langle\langle L\rangle\rangle &= \max\deg\langle\langle L\rangle\rangle - \min\deg\langle\langle L\rangle\rangle \\
&= V+2(N-1)-(-V-2(M-1)) \\
&= V+2N-2+V+2M-2 \\
&= 2V+2(N+M)-4 \\
&= 2V+2(R)-4 = 2V+2(V+2)-4 = 4V
\end{aligned}
$$

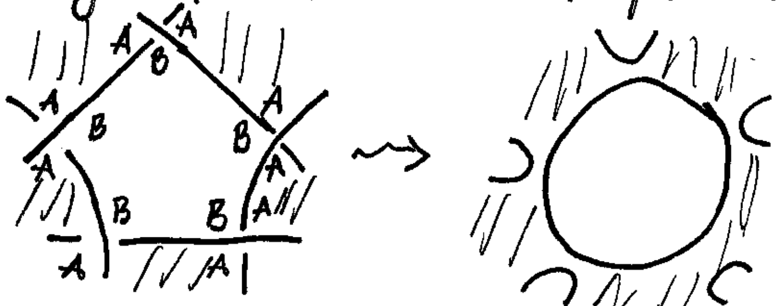
Proof of main proposition: Recall state sum formula

$$\langle\langle L \rangle\rangle = \sum_{S} A^{\alpha(S)} A^{-\beta(S)} (-A^2 - A^{-2})^{\gamma(S)-1}$$

Let S_0 be the state with all A-splits



Then $\alpha(S_0) = V$ $\beta(S_0) = 0$, We claim $\gamma(S_0) = N$, the number of unshaded regions. This because A-splits never connect B's:



So the term for S_0 is $A^V(-A^2 - A^{-2})^{N-1}$, and has highest power

$$A^V (A^2)^{N-1} = A^{V+2(N-1)}$$

Any other state is obtained from S_0 by changing some A-splits to B-splits. That decreases α by 1, increases β by 1, and changes γ by ± 1. Thus the exponent in

$A^{\alpha} A^{-\beta}$ goes down by 2, and the highest exponent in $(-A^2 - A^{-2})^{\gamma - 1}$ can go up by at most 2. So overall highest exponent cannot increase when we change A-split to B-split. Furthermore, the highest term from S_0 cannot be canceled by any other state: If S_1 is obtained from S_0 by switching an A to B, Then $\gamma(S_1) < \gamma(S_0)$. For otherwise, some unshaded region would touch both sides of a crossing, contradicting simplicity. The rest of the proposition is proved by considering the state with all B-splits.

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Last time: If L is an alternating link diagram that is simple with V crossings, then $V = \frac{\text{span}\langle\langle L\rangle\rangle}{4}$.

Since $\text{span}\langle\langle L\rangle\rangle = \text{span}\,X(L)$ is invariant, any two simple alternating diagrams for the same link have the same number of crossings.

We can generalize this to

Theorem: For any link diagram L with V crossings,

$$V \geq \frac{\text{span}\langle\langle L\rangle\rangle}{4}$$

Corollary: A simple alternating diagram [redacted] for L (if it exists) has the fewest number of crossings.

The proof of the theorem relies on a lemma:

Dual State Lemma: Let L be a (connected) link diagram. Let S be a state (collection of A/B-splits), and denote by $\hat{S}$ the state with all splits switched (the "dual state")

Then $\gamma(S) + \gamma(\hat{S}) \leq R$ = # of regions.

Problem: Prove the Dual State Lemma.

Hint: induction on V.

<u>Proof of theorem</u>: let L be any Link diagram with V crossings. Let S_0 be the state with all A-splits.

This contributes $A^{\alpha(S_0)} A^{-\beta(S_0)} (-A^2-A^{-2})^{\gamma(S_0)-1}$

$$= A^V A^0 (-A^2-A^{-2})^{\gamma(S_0)-1} = \pm A^{V+2(\gamma(S_0)-1)} \text{ + lower powers of } A$$

Again, each time we change an A-split to a B-split, The overall highest exponent [redacted]

$\alpha(s)-\beta(s)+2(\gamma(s)-1)$ cannot increase.

So we can conclude $\text{maxdeg} \langle\langle L\rangle\rangle \leq V+2(\gamma(S_0)-1)$

Now consider $\hat{S}_0$, the dual state, that has all B-splits.

It contributes $A^0 A^{-V} (-A^2-A^{-2})^{\gamma(\hat{S}_0)-1} = \pm A^{-V-2(\gamma(\hat{S}_0)-1)}$ + higher powers of A

each time we change a B-split to an A-split, the lowest power cannot decrease, so

$$\text{min deg} \langle\langle L\rangle\rangle \geq -V-2(\gamma(\hat{S}_0)-1)$$

$$-\text{min deg} \langle\langle L\rangle\rangle \leq V+2(\gamma(\hat{S}_0)-1)$$

Therefore $\text{span} \langle\langle L\rangle\rangle \leq V+2(\gamma(S_0)-1)+V+2(\gamma(\hat{S}_0)-1)$

$$= 2V+2(\gamma(S_0)+\gamma(\hat{S}_0))-4$$

by dual state Lemma $\leq 2V+2R-4 = 2V+2(V+2)-4$

$$= 4V.$$

July 19, 2019

More Knot polynomials:

Skein relations: suppose L_+, L_-, L_0 are oriented link diagrams that differ only in a small disk where they are related by

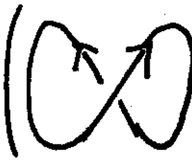
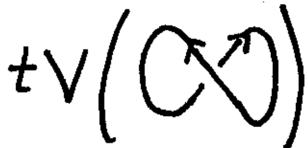
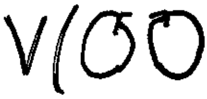
Then $\quad A^4 X(L_+) - A^{-4} X(L_-) = (A^{-2} - A^2)\, X(L_0)$

If we use the variable $t = A^{-4}$, This becomes

$$t^{-1} V(L_+) - t V(L_-) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V(L_0)$$

Where $V(L)$ is $X(L)$ with A replaced by $t^{-1/4}$.

It's possible to compute $V(L)$ using just this relation and the fact that $V(L)$ is invariant (and $V(\bigcirc) = 1$).

Eg $\; t^{-1} V\Big($  $\Big) - t V\Big($  $\Big) = (t^{1/2} - t^{-1/2})\, V\Big($  $\Big)$

$t^{-1} \cdot 1 - t \cdot 1 = (t^{1/2} - t^{-1/2})\, V($  $)$

$\therefore V($  $) = \dfrac{t^{-1} - t}{t^{1/2} - t^{-1/2}} = -t^{1/2} - t^{-1/2}$

$t^{-1} V\Big($  $\Big) - t V\Big($  $\Big) = (t^{1/2} - t^{-1/2})\, V\Big($  $\Big)$

$t^{-1} V($  $) - t(-t^{1/2} - t^{-1/2}) = t^{1/2} - t^{-1/2}$

$V($  $) = -t^{1/2} - t^{5/2}$

After Jones polynomial came out, many mathematicians jumped on the problem of finding the most general knot polynomial invariant that is determined by a skein relation. The result is called the HOMFLY polynomial. It is a polynomial in positive and negative powers of l and m.

Theorem There is a unique way to associate a polynomial $P(L)$ to each oriented link L so that

$$P(\bigcirc)=1 \qquad l P(L_+)+l^{-1}P(L_-)+mP(L_0)=0$$

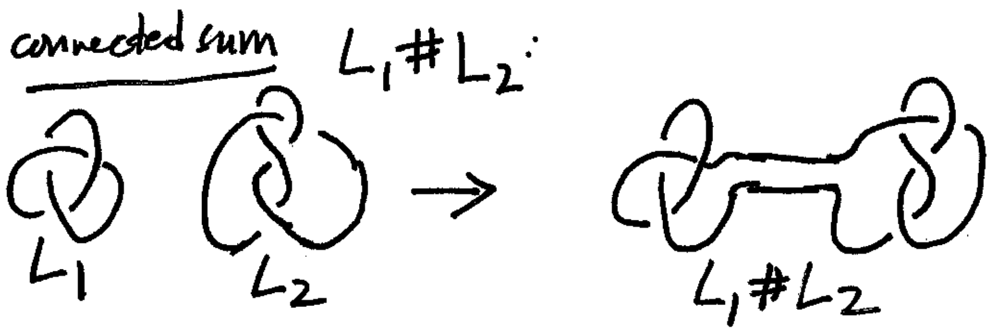
Problem: Compute the HOMFLY polynomial of the trefoil .

The Jones polynomial can be gotten from the Homfly polynomial By setting $l=it^{-1}$, $m=i(t^{-1/2}-t^{1/2})$ $(i=\sqrt{-1})$

If we set $l=i$, $m=i(t^{1/2}-t^{-1/2})$, we get the so-called Alexander polynomial $\Delta(L)$. This was actually discovered by Alexander in 1928.

Problems: Show that $P(L \cup \bigcirc)=-(l+l^{-1})m^{-1}P(L)$

Define connected sum $L_1 \# L_2$:



Show $P(L_1 \# L_2)=P(L_1)P(L_2)$.

YSP Knots July 22, 2019

Kauffman's interpretation of the Jones polynomial in terms of the bracket $\langle L \rangle$ and the state sum formula is a hint at a broader connection between knots, graphs, and general "sums over states."

There is an area of science that is entirely based around such "sums over states" called <u>Statistical mechanics</u>.

In statistical mechanics, we consider a <u>system</u> consisting of <u>particles</u> that can be in various <u>states</u>, and which <u>interact</u> somehow. We are interested in the average properties of the system, but the actual microscopic state of the system behaves randomly.

Stat. Mech. provides a theoretical foundation for thermodynamics, as it should allow us to compute things like the melting/boiling points of different substances, or derive laws like

$$PV = nRT$$

In mathematics we often look at highly simplified models.

Reference: Ideal Gas Constant $R = 8.314\ J/K\cdot mol$

Avogadro's Number $N_A = 6.022 \times 10^{23}$ particles/mole

Boltzmann's constant $k = R/N_A = 1.38 \times 10^{-23} J/K$

Postulates of Statistical mechanics

1. The system can be in any state s, with some probability $p(s)$. $p(s)$ can be a number between 0 and 1, and the sum of all probabilities is 1

$$\sum_{\substack{\text{states}\\ s}} p(s) = 1$$

2. If $f(s)$ is some observable quantity (e.g. Energy) then the average (or expected) value of f is

$$\langle f \rangle = \sum_{\substack{\text{states}\\ s}} f(s)\,p(s) \quad \text{(weighted average)}$$

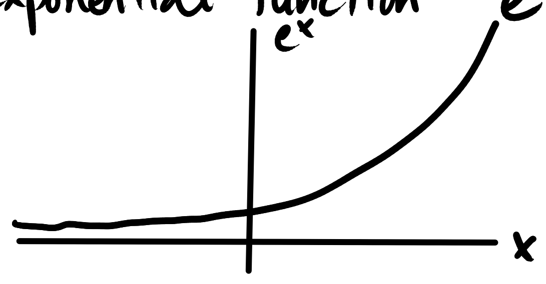
3. There is a function $E(s)$ called the <u>energy</u>, whose form expresses all of the interactions in the system.

4. (Boltzmann-Gibbs Law) The probability of the system being in state s is proportional to the exponential of $E(s)$, more specifically

$$p(s) \propto e^{-E(s)/kT}$$

where k is Boltzmann's constant and T is the temperature.

In 4, we are using the exponential function e^x, where $e = 2.718281828\ldots$



Some important facts about exponentials July 21, 2019

- $e^0 = 1$, $e^{a+b} = e^a e^b$, $e^{-a} = \frac{1}{e^a}$, $(e^a)^b = e^{ab}$
- As x goes to ∞, e^x goes to ∞.
- As x goes to $-\infty$, e^x goes to 0.

Postulates 1 and 4 imply that there is a constant C such that $p(s) = C e^{-E(s)/kT}$ and

$$1 = \sum_s p(s) = \sum_s C e^{-E(s)/kT} = C \sum_s e^{-E(s)/kT}$$

so $C = \left(\sum_s e^{-E(s)/kT} \right)^{-1}$. By convention we write

$$Z = \sum_c e^{-E(s)/kT}, \text{ so } C = Z^{-1}$$

Z is called the "partition function", and most thermodynamic properties of the system can be derived from it.

Eg.
Total Energy $\langle E \rangle = \sum_s E(s) p(s) = \dfrac{\sum_s E(s) e^{-E(s)/kT}}{Z}$

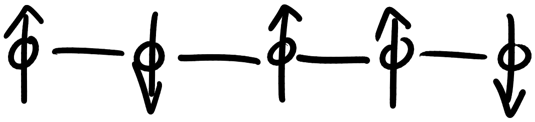
Free Energy $F = -kT \ln Z$

$(\langle E \rangle - F)/T$ is the Entropy of the system.

Example Ising model: This is a model of a magnet. We have a number N of molecules arranged on a line.

Each molecule behaves like a microscopic magnet (spin) and can point either up or down. Thus there are 2^N total states.

e.g.:



Each molecule interacts with it's nearest neighbors, and it "wants" to be aligned with them. Number the sites 1 thru N, so that i neighbors $i-1$ and $i+1$. Set $s_i = \begin{cases} +1 & \text{if spin at } i \text{ is up} \\ -1 & \text{if spin at } i \text{ is down.} \end{cases}$

$$\text{Set } E(s_i, s_{i+1}) = \begin{Bmatrix} -J & s_i = s_{i+1} \\ J & s_i \neq s_{i+1} \end{Bmatrix} = -J s_i s_{i+1}$$

where $J > 0$ is a constant.

$$E(s) = E(s_1, s_2, \ldots, s_N) = \sum_{i=1}^{N-1} E(s_i, s_{i+1}) = -J \sum_{i=1}^{N-1} s_i s_{i+1}$$

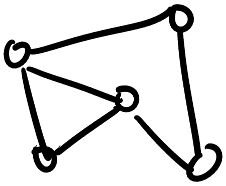
The two states with the greatest probability are those where all spins are aligned (all up or all down) since these have the lowest Energy $E = -J(N-1)$

Q: What is the probability that there is one spin up and all others down?

YSP Knots July 23, 2019

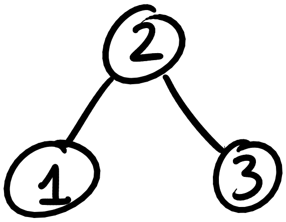
The so-called Potts model gives a very nice connection between statistical mechanics and graph topology.

Recall: A graph is a network of vertices joined by edges.



We can number the vertices with $1, 2, 3, \ldots, N$, and then we denote the edge joining i to j by (i, j).

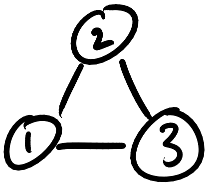
It is sometimes helpful to think in terms of abstract sets



$V = \{1, 2, 3\}$ vertices

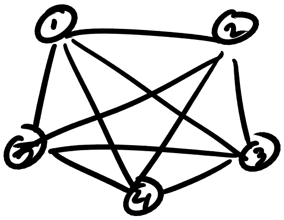
$E = \{(1,2), (2,3)\}$ edges

or



$V = \{1, 2, 3\}$

$E = \{(1,2), (2,3), (1,3)\}$



$V = \{1, 2, 3, 4, 5\}$
$E = \{\text{all pairs}\}$
] called a complete graph.

Colorings: Pick a whole number q, and consider a set of q colors (eg $q=3$, red, green, blue)

A vertex coloring of graph G is an assignment of a color to each vertex. A vertex coloring is proper if whenever two vertices are joined by an edge, the have different colors.

A graph is called q-colorable if it admits a proper vertex coloring with q colors.

Ex. Bipartite = 2-colorable

4-color theorem (Appel-Haken 1977)
Every planar graph is 4-colorable.

For a graph G, define $C_G(q) = \#$ of proper coloring with q colors

G is q-colorable $\Leftrightarrow$ $C_G(q) > 0$.

How to calculate $C_G(q)$?

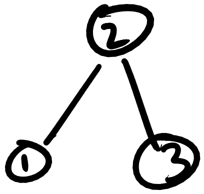
Potts model: Take graph G. We have a molecule at each vertex, which is allowed to be in one of q spin states $s_i = 1, 2, \dots, q$. So there are q^N total states.

Two molecules interact if there is an edge between them. The interaction energy is $-J\,\delta(s_i, s_j)$

Where $\delta(s_i, s_j) = \begin{cases} 1 & \text{if } s_i = s_j \\ 0 & \text{if } s_i \neq s_j \end{cases}$ July 23, 2019

Total Energy $E(s) = -J \sum_{\substack{\text{Edges} \\ (i,j)}} \delta(s_i, s_j)$

Eg.



$E(s) = -J\left(\delta(s_1, s_2) + \delta(s_2, s_3)\right)$

Partition function: $Z = \sum_{\substack{\text{states} \\ s}} e^{-E(s)/kT}$

$$= \sum_{\substack{\text{states} \\ s}} e^{\left\{\frac{J}{kT} \sum_{\substack{\text{edges} \\ (i,j)}} \delta(s_i, s_j)\right\}} = \sum_{\substack{\text{states} \\ s}} \prod_{\substack{\text{edges} \\ (i,j)}} e^{\frac{J}{kT}\delta(s_i, s_j)}$$

$$= \sum_{\substack{\text{states} \\ s}} \prod_{\substack{\text{edges} \\ (i,j)}} \left(e^{J/kT}\right)^{\delta(s_i, s_j)}$$

Observe $\left(e^{J/kT}\right)^{\delta(s_i, s_j)} = \begin{cases} e^{J/kT} & \text{if } s_i = s_j \\ 1 & \text{if } s_i \neq s_j \end{cases}$

Now suppose $J < 0$, so it is energetically favorable to have $s_i \neq s_j$ when (i,j) is an edge.

Take $T \rightarrow 0$ (put it in the freezer)

Then $e^{J/kT}$ becomes very close to 0.

So in the state sum for Z, a term is very small if it has even 1 pair s_i, s_j such that (i,j) is an edge and $s_i = s_j$.

The terms that dominate are those that correspond to <u>proper</u> vertex colorings. These terms have the value 1. Thus:

If $J<0$ and $T \to 0$, $Z = C_G(q)$
is the number of proper colorings with q colors.

We can rewrite things slightly. Set $v = e^{J/kT} - 1$

then $\left(e^{J/kT}\right)^{\delta(s_i,s_j)} = (1+v)^{\delta(s_i,s_j)} = 1 + v\,\delta(s_i,s_j)$

Thus $$Z = \sum_{\substack{\text{states} \\ S}} \prod_{\substack{\text{edges} \\ (i,j)}} \left[1 + v\,\delta(s_i,s_j)\right]$$

Taking $J<0$ and $T \to 0$ corresponds to setting $v=-1$.

We may write $Z = Z_G(q,v)$ to express dependence on, Graph G, number of colors q, and v.

Thus $$C_G(q) = Z_G(q,-1) = \sum_{\substack{\text{states} \\ S}} \prod_{\substack{\text{edges} \\ (i,j)}} \left[1 - \delta(s_i,s_j)\right]$$

<u>Problem</u>: Compute $Z_G(q,v)$ for some simple graphs
Show that it is a polynomial function of q and v
(either in your examples or in general)

YSP Knots July 24, 2019

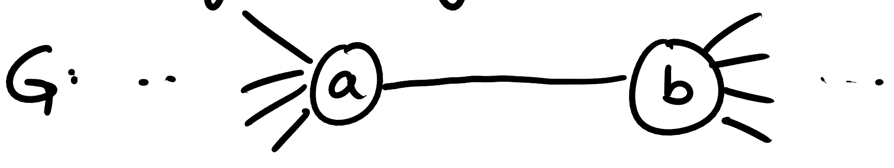
Last time: Potts model on a graph G
We have q possible spins/colors at each vertex.
Partition function is:

$$Z_G = \sum_{\substack{\text{states}\\ s}} \prod_{\substack{\text{edges}\\ (i,j)}} \left(e^{J/kT}\right)^{\delta(s_i, s_j)} = \sum_{\substack{\text{states}\\ s}} \prod_{\substack{\text{edges}\\ (i,j)}} \left[1 + v\,\delta(s_i, s_j)\right]$$

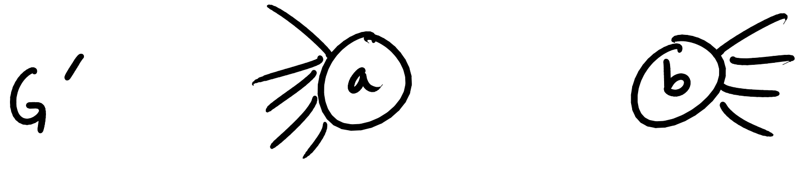
where $v = e^{J/kT} - 1$.
It turns out that Z_G is a polynomial function of the number of colors q and v.
Called the <u>dichromatic polynomial of G</u>.

One way to calculate it and show it is a polynomial is via a recursion:

Take a graph G, and suppose vertices a and b are joined by an edge



Now let G′ be G with this edge deleted



How does Z_G differ from $Z_{G'}$?

$$Z_G = \sum_{\substack{\text{states}\\ S}} \prod_{\substack{\text{edges}\\ (i,j)}} \left[1 + v\,\delta(s_i, s_j)\right]$$

$$= \sum_{\substack{\text{states}\\ S}} \left[1+v\,\delta(s_a,s_b)\right] \prod_{\substack{\text{edges}\\ (i,j)\\ \text{except}\\ (a,b)}} \left[1+v\,\delta(s_i,s_j)\right]$$

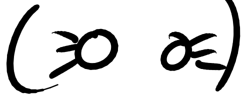
$$= \left\{\sum_{\substack{\text{states}\\ s}} \prod_{\substack{\text{edges}\\ (i,j)\\ \text{except}\\ (a,b)}} \left[1+v\,\delta(s_i,s_j)\right]\right\}$$

$$+ v\left\{\sum_{\substack{\text{states}\\ \text{where}\\ s_a=s_b}} \prod_{\substack{\text{except}\\ (a,b)}} \left[1+v\,\delta(s_i,s_j)\right]\right\}$$

The first term is $Z_{G'}$ and the second is $v Z_{G''}$ where G'' is G with the edge (a,b) contracted.

G'' :

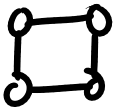
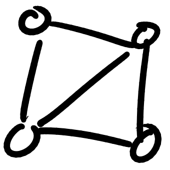
So $Z_G = Z_{G'} + v Z_{G''}$

or $Z($  $) = Z($  $) + v\,Z($  $)$

July 24, 2019

This is the deletion-contraction recursion for the dichromatic polynomial.
Note that both G′ and G″ have 1 fewer edge than G has. So this rule lets us recursively reduce to graphs with fewer edges.
In the end, we reduce to a graph with no edges.
Then we can use

$$Z\Big(\underbrace{\circ\ \circ\ \ \circ\cdots\circ}_{\substack{V \text{ vertices}\\ \text{No edges}}}\Big) = q^V$$

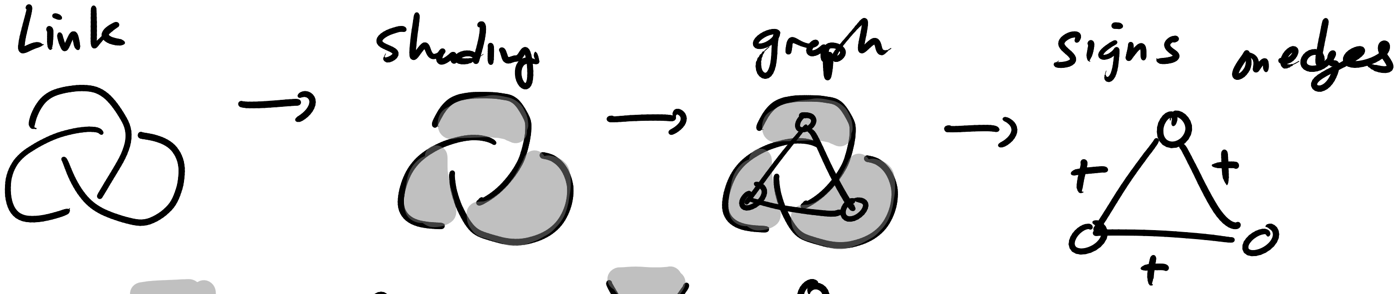
Problem: compute Z() and Z()

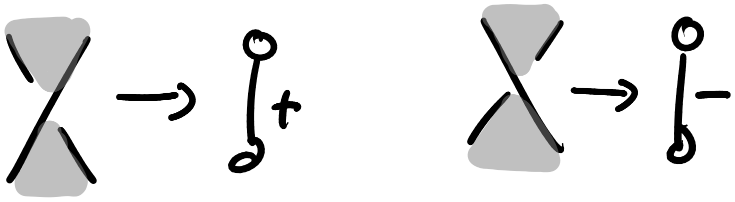
This recursion is similar to Kauffman bracket rules

$$\langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$$

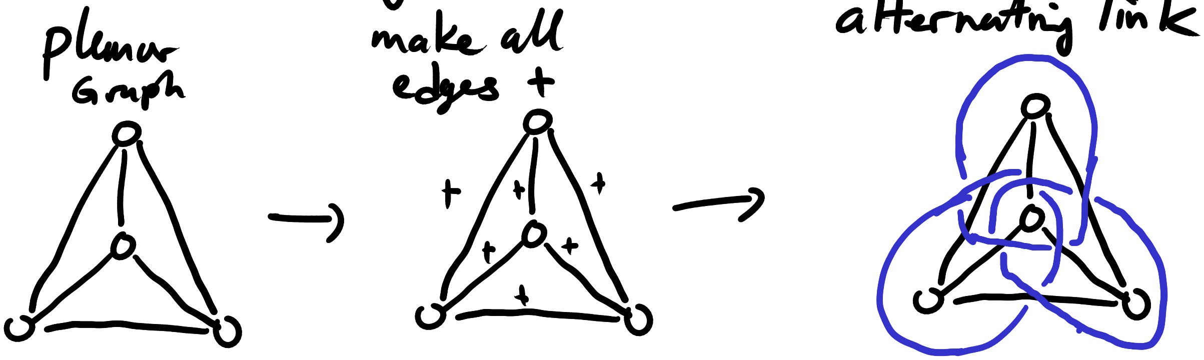
$$\langle N \text{ disjoint circles} \rangle = C^{N-1}$$

Recall the medial graph construction.

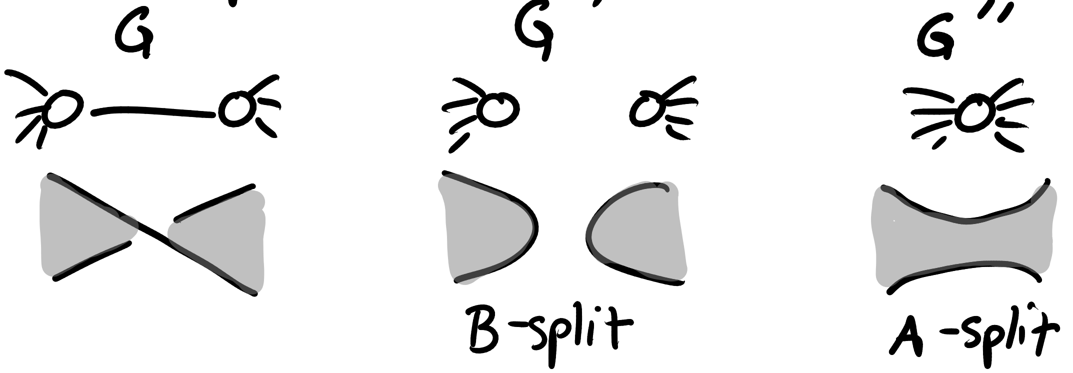




Conversely given any planar graph G, we can construct an alternating link.



Now compare



This motivates:

Definition: The <u>dichromatic Kauffmann bracket</u> $[L]$ of an alternating link diagram L (WARNING: this isn't a link invariant) is defined by the rules

(i) $[\times] = q^{-1/2} v [\,)(\,] + [\asymp]$

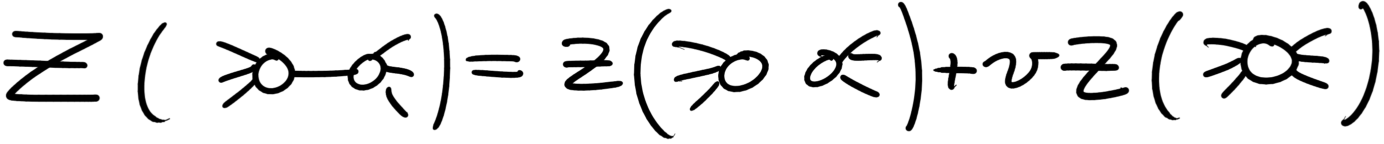
(ii) $[\,O\ \boxed{L}\,] = q^{1/2}[L]$

(iii) $[O] = q^{1/2}$

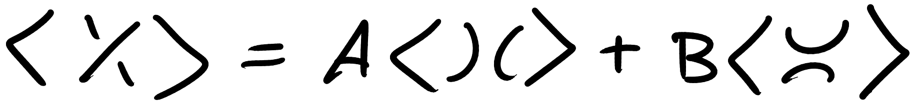
Theorem: If G is a graph and L is the corresponding alternating link diagram: $Z(G) = q^{V/2}[L]$ $V = \#$ vertices in G

YSP Knots July 25, 2019

The analogy between the deletion-contraction recursion for the dichromatic polynomial



and the Kauffman bracket recursion

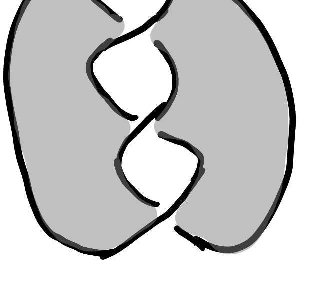


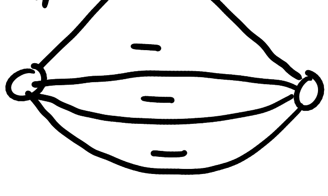
Together with the fact that $Z(G)$ is the partition function of the Potts model

$$Z(G) = \sum_{s} \prod_{(i,j)} [1 + v\,\delta(s_i, s_j)]$$

suggests that a modification of the Potts model could give rise to the Kauffman bracket or Jones polynomial itself.

Link diagram and corresponding signed planar graph:





Generalized Potts model: We still have a spin/color at each vertex $s_i = 1, 2, \dots, q$.

We replace the factor $(e^{J/kT})^{\delta(s_i, s_j)} = 1 + v\,\delta(s_i, s_j)$ with another function of s_i and s_j. We can actually use a different function depending on whether the edge is + or −. Then we have

$$Z = \sum_{\substack{\text{states} \\ S}} \prod_{+\text{edges}} w_+(s_i, s_j) \prod_{-\text{edges}} w_-(s_i, s_j)$$

where $w_+(s_i, s_j)$ and $w_-(s_i, s_j)$ are the weights for + and − edges respectively. We need to determine these. Some reasonable assumptions are

$w_\pm$ are symmetric: $w_+(a,b) = w_+(b,a)$
$w_-(a,b) = w_-(b,a)$

$w_\pm(a,b)$ only depends on whether $a=b$ or not.

$$w_+(a,b) = \begin{Bmatrix} w_{+,=} & \text{if } a=b \\ w_{+,\neq} & \text{if } a\neq b \end{Bmatrix} \quad w_-(a,b) = \begin{Bmatrix} w_{-,=} & \text{if } a=b \\ w_{-,\neq} & \text{if } a\neq b \end{Bmatrix}$$

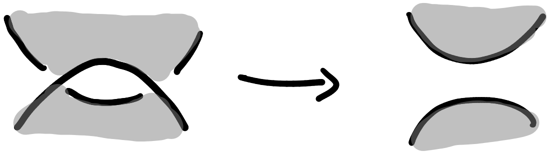
So we really get to choose 4 numbers (which may depend on q.)

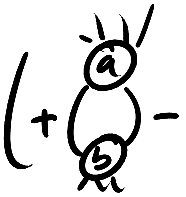
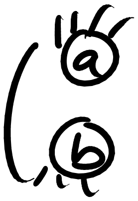
Let V = # vertices of graph = # crossings of L

We seek a set of weights that makes $q^{-V/2} Z$ invariant under Reidemeister II and III moves.

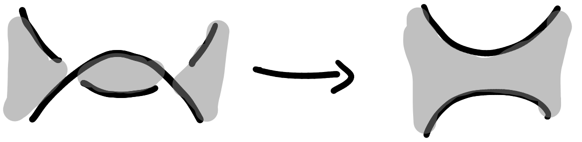
What equations must w_I satisfy. July 25, 2019

Reidemeister II:

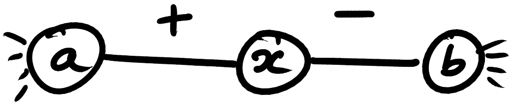
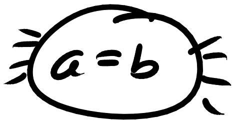


Term() = $w_+(a,b) w_-(a,b)$ Term()

Works if $w_+(a,b)\, w_-(a,b) = 1$ for all a, b.



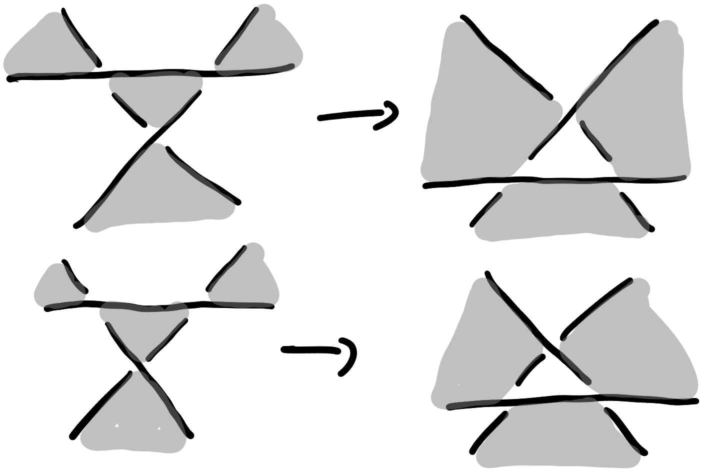
[Note this reduces V by 2.]

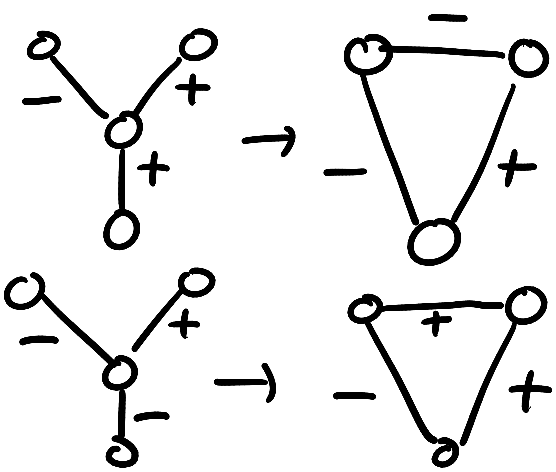
 vs 

This works if $\sum_{x=1}^{q} w_+(a,x)\, w_-(x,b) = q\, \delta(a,b)$

[The q factor is there so that $q^{-V/2}Z$ is unchanged]

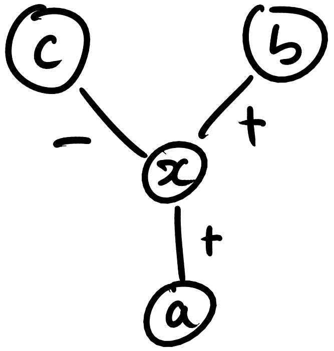
Reidemeister III = Star-Triangle exchange.



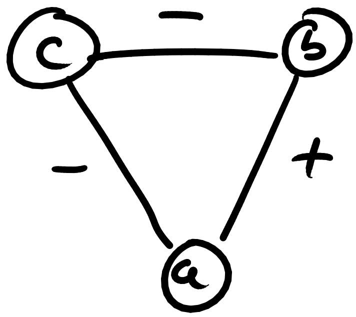


[Note this reduces V by 1]

This will work if summing over the spins at the center of the star gives the same result as the other picture.



vs



$$\sum_{x=1}^{q} w_+(a,x)\, w_+(b,x)\, w_-(c,x) \qquad (\text{for all } a,b,c)$$

$$= \sqrt{q}\, w_+(a,b)\, w_-(b,c)\, w_-(a,c)$$

[Note $\sqrt{q}$ is so that $q^{-V/2}Z$ is unchanged]

Also need same equation with +/− flipped.
Can these equations be satisfied? Amazingly, yes.
Choose t so that $q = 2+t+t^{-1} = (t^{1/2}+t^{-1/2})^2$

Then set $w_+(a,b) = \begin{cases} i t^{-3/4} & a=b \\ -i t^{1/4} & a \neq b \end{cases}$

$$w_-(a,b) = \begin{cases} -i t^{3/4} & a=b \\ i t^{-1/4} & a \neq b \end{cases}$$

I claim that with these weights, for a link diagram L

$$Z(L) = (-1)^{\#\text{components}(L)-1} \left(i t^{-3/4}\right)^{wr(L)} \left(t^{1/2}+t^{-1/2}\right)^{V+1} V_L(t)$$

where $V_L(t)$ is the Jones polynomial of L.
($X(L)$ with substitution $A = t^{-1/4}$)